Exercise 1. ***The Motion of Vortices***

Consider a single two-dimensional vortex of strength ω placed in the origin. It produces a 2D velocity field given by

$$\mathbf{u}(\mathbf{r}) = \frac{\omega}{r}\mathbf{e}_\theta,$$

which can be rewritten in components as

$$u_x = -\frac{\omega y}{r^2},\ u_y = \frac{\omega x}{r^2}.$$

In this problem we will study the motion of many such vortices in their own velocity field. Suppose that we have n vortices of the same strength ω at distinct positions $\mathbf{r}_i = (x_i, y_i)$.

(a) Explain why the velocity of the i^{th} vortex is given by

$$\dot{x}_i = -\omega \sum_{j\neq i} \frac{y_i - y_j}{(x_i - x_j)^2 + (y_i - y_j)^2},\ \dot{y}_i = \omega \sum_{j\neq i} \frac{x_i - x_j}{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{1}$$

Quite surprisingly, this system can be described by using the Hamiltonian formalism. For each vortex we now define a generalized coordinate $q_i = x_i$ and identify $p_i = y_i$ with its 'canonical momentum'.

(b) Show that the equations (1) are equivalent to the Hamilton's equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k},\ \dot{p}_k = -\frac{\partial H}{\partial q_k} \text{ with } H = -\frac{\omega}{4}\sum_{\substack{i,j\\ i\neq j}} \log\left[(q_i - q_j)^2 + (p_i - p_j)^2\right]. \tag{2}$$

Hint: After differentiating H it is convenient to write the sum as

$$\frac{\partial H}{\partial q_k} = \sum_{j\neq i}\sum_i A_{ij}\delta_{ik} + \sum_{i\neq j}\sum_j B_{ij}\delta_{jk}$$

for some objects A_{ij}, B_{ij} and use the properties of the Kronecker δ_{ij}.

(c) Consider an infinitesimal rotation of the system given by

$$\delta q_i = -\epsilon p_i,\ \delta p_i = \epsilon q_i,\ i = 1, 2, ..., n \tag{3}$$

and show that it does not change the Hamiltonian to the order $\mathcal{O}(\epsilon)$. What is the generating function $\tilde{G}(q_1, ..., q_n, p_1, ..., p_n)$ of such transformation?

Hint: You should find the expansion $\log(1 + \epsilon^2) = \epsilon^2 + \mathcal{O}(\epsilon^4)$ useful. Recall that the generating function is defined by $\delta p_i = -\epsilon\partial\tilde{G}/\partial q_i$, $\delta q_i = \epsilon\partial\tilde{G}/\partial p_i$.

(d)* Define new coordinates $Q_i(\epsilon) = q_i + \delta q_i$, $P_i(\epsilon) = p_i + \delta p_i$ and the transformed Hamiltonian $H(\epsilon) = H(Q_i(\epsilon), P_i(\epsilon))$. Use the result from c) to deduce the value of $\mathrm{d}H/\mathrm{d}\epsilon|_{\epsilon=0}$. Hence show that $\tilde{G}$ is a conserved quantity, i.e. $\mathrm{d}\tilde{G}/\mathrm{d}t = [\tilde{G}, H] = 0$.

Exercise 2. ***One more exercise about Canonical transformations...***

Given the transformation for a system of one degree of freedom

$$\begin{aligned} Q &= q\,\cos\alpha\,-\,p\,\sin\alpha \\ P &= q\,\sin\alpha\,+\,p\,\cos\alpha \end{aligned} \tag{4}$$

(a) show that it satisfies the symplectic condition for any value of the parameter α.

(b) Find a generating function F for the transformation.
Hint: you can, for example, chose a generating function which depends on Q and q, so that $p = p(Q, q)$ and $P = P(Q, q)$, and use the transformation equations:

$$p = \frac{\partial F}{\partial q} \quad \text{and} \quad P = -\frac{\partial F}{\partial Q}\,. \tag{5}$$

(c) What is the physical significance of the transformation for $\alpha = 0$? For $\alpha = \pi/2$?
Does your generating function work for both of these cases?

(d) * If not, can you find a generating function valid for the case where the one you have just found doesn't work?
Hint: try with a generating function which depends on other variables, e.g. q and P.

Exercise 3. ***Conserved tensor for harmonic oscillator***

We have seen that the Kepler problem features, in addition to the total energy and angular momentum, a conserved quantity known as Laplace-Runge-Lenz vector. It turns out that the existence of conserved Laplace-Runge-Lenz-like vectors are not specific to the Kepler potential. Instead, it is rather a feature of radial force problems.

As such it is not surprising that also the three-dimensional isotropic harmonic oscillator allows for such a conserved quantity, which in this case is the (six component) tensor

$$\mathbf{A}_{ij} = \frac{1}{\lambda}(p_i p_j + \lambda^2 r_i r_j)\,, \tag{6}$$

where $i, j \in \{x, y, z\}$.

(a) Consider a particle of mass m in the harmonic potential

$$V(\vec{r}) = \frac{1}{2}u\vec{r}^{\,2}\,, \qquad \vec{r} \in \mathbb{R}^3. \tag{7}$$

Write down the Lagrangian and find the Hamiltonian from the Legendre transformation.

(b) Using Poisson brackets $[\cdot,\cdot]$ and a suitable choice of λ, show that the tensor $\mathbf{A}_{ij}$ defined in Eq. (6) is conserved, i.e.,

$$[\mathbf{A}_{ij}, H] = 0\,, \qquad \forall\, i, j\,. \tag{8}$$

Hint. Although the problem has radial symmetry, note that it might be simpler to carry out the calculation in cartesian coordinates.

(c) The tensor $\mathbf{A}_{ij}$ has six components, corresponding to six constants of motion. The energy and the angular momentum are additional constants of motion, because the harmonic oscillator is a central force problem. Hence there are more constants of motion than degrees of freedom. For this reason some of them must be interlinked.

In this context, relate

$$\operatorname{tr} \mathbf{A}_{ij} = \sum_i \mathbf{A}_{ii} \tag{9}$$

to another conserved quantity.