Exercise 1. ***From Coupled Oscillators to Wave Equation***

Consider a ring of N masses m free to oscillate about their respective equilibrium positions, coupled by springs of spring constant κ. Let ϕ_k denote the displacement of the k-th mass from its equilibrium position. We impose periodic boundary conditions, i.e.: $\phi_{k+N} = \phi_k$ and *neglect the curvature of the chain*, such that it behaves as a *1D chain of masses* where the last mass couples to the first.

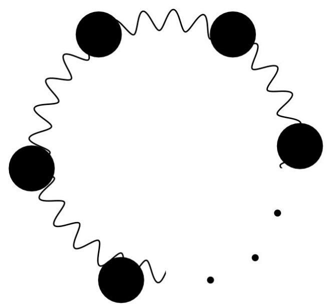

Figure 1: A ring of N masses

i) Show that the Lagrangian of the system is given by:

$$\mathcal{L} = \sum_{k=1}^{N} \left(\frac{1}{2} m \dot{\phi}_k^2 - \frac{1}{2} \kappa (\phi_k - \phi_{k+1})^2 \right)$$

Be especially careful not to overcount the potential terms twice, i.e. note that:

$$\sum_{k=1}^{N} \frac{1}{2} \kappa ((\phi_k - \phi_{k+1})^2 + (\phi_k - \phi_{k-1})^2) = \sum_{k=1}^{N} \kappa (\phi_k - \phi_{k+1})^2$$

since you can shift the indices under the sum thanks to the boundary conditions (convince yourself about it).

ii) Convince yourself that the equation of motion for the k-th mass is given by:

$$\ddot{\phi}_k = \frac{\kappa}{m} (\phi_{k-1} - 2\phi_k + \phi_{k+1})$$

If in doubt, use force!

iii) Make an ansatz for the solution of the form $\phi_k \propto \exp(i(\varsigma k + \omega t))$, where ς, ω are constants that can be interpreted as wavenumber and (angular) frequency respectively (why?). Find the expression for ω by inserting your ansatz into the equations. You should find:

$$\omega = \pm 2 \sqrt{\frac{\kappa}{m}} \left| \sin\left(\frac{\varsigma}{2}\right) \right|$$

Note 1: This tells you that for every choice of ς, there are two ω frequency solutions (with the exception of $\varsigma = 0$ for which the solutions are degenerate), one traveling to the "left" and the other to the "right". You should give an argument for what the "traveling" means.

iv) By recalling that we are dealing with a string of masses and therefore $\phi_{k+N} = \phi_k$, convince yourself that there is a countably finite set of allowed ς.

Note 2: These correspond to different normal frequencies of your chain - and each ansatz function with such allowed ς corresponds to a normal mode on your chain.

v) Consider $N \to \infty$, while keeping the overall circumference of the mass ring constant. In such case $a \to 0$ and one can approximate the index k by a continuous position "label" $x = ka$: $\phi_k(t) \to \phi(ka, t) = \phi(x, t)$. Convince yourself that the equation of motion now reads:

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = \frac{\kappa}{m}(\phi(x-a,t) - 2\phi(x,t) + \phi(x+a,t))$$

iv) Taylor expand $\phi(x \pm a)$ around x to the second order to show that:

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = \frac{\kappa a^2}{m}\frac{\partial^2 \phi(x,t)}{\partial x^2}$$

Consider a case where $\kappa \to \infty$ as we go to $a \to 0$, in a way that $\frac{\kappa a^2}{m} = const. = \frac{1}{c^2}$. Have you seen such equation before? What is the interpretation of the constant c? Can you justify your motivation by some clever argument? If you are keen enough, check that $f(x \pm ct)$ is a solution to this equation for any sufficiently smooth function f.

Note 3: If you cannot recall the equation, remind yourself what a wave equation looks like, as you will use it (with some damping/dissipative term) in one of the following exercises.

Note 4: The coupled harmonic oscillator chain is a standard way to give a motivation for a notion of field. As you will see in your advanced courses such as General Relativity or Quantum Field Theory, the field viewpoint provides enormous insight into a lot physical phenomena.

Exercise 2. *Moments of inertia*

The Huygens-Steiner theorem (sometimes also known as "parallel axis theorem") is an easy-to-prove result that relates moments of inertia for rotations of the same rigid body around any axis to the one around a parallel axis through its center of mass.

a) Let $\mathcal{I}_a$ be the moment of inertia of a rigid body with mass m for rotations around an axis a that goes through its center of mass. Show that the moment of inertia $\mathcal{I}_b$ for rotations around any axis b parallel to a can be obtained from $\mathcal{I}_a$ through the formula

$$\mathcal{I}_b = \mathcal{I}_a + MR^2, \tag{1}$$

where R is the distance between a and b.

b) Compute the moment of inertia for a uniform ball of mass M and radius R with respect to an axis that goes through its center. Then use the Huygens-Steiner theorem to see how the result changes when the ball rotates around an axis at $R/2$ from its center.

c) Compute the moment of inertia for a thin spherical shell of mass M and radius R with respect to an axis passing through its center. Compare with the previous result.

d) Compute the moment of inertia for a spherical cloud of mass M and density proportional to e^{-r}, always with respect to an axis through its center.

Exercise 3. ***Oscillating string with friction.***

A uniform string has length L and mass per unit length ρ. It undergoes small transverse vibration in the (x, y) plane with its endpoints held fixed at $(0, 0)$ and $(L, 0)$ respectively. The tension is K. The string is subject to a small velocity-dependent frictional force $-kv\delta l$ to each small piece of length δl with transverse velocity v. Using appropriate approximations, the following equations hold for the vibration amplitude $y(x, t)$:

$$\frac{\partial^2 y}{\partial t^2} + a\frac{\partial y}{\partial t} = b\frac{\partial^2 y}{\partial x^2} \tag{2}$$

$$y(0, t) = 0 = y(L, t) \tag{3}$$

(a) Find the constants a and b in (2).

(b) Find all solutions of (2) and (3) which have the product form $y = X(x)T(t)$. You may assume $a^2 < b/L^2$.
(Hint: The wave equation (2) is separable, i.e. it can be written as $F(X'', X', X, x) = G(T'', T', T, t)$. In order for the equation to hold for all x and t, each side must be equal to a constant, which is taken as $-\lambda^2$. Solve the two equations for $T(t)$ and $X(x)$ separately, using the boundary conditions (3).)