Exercise 1. ***Laplace-Runge-Lenz vector***

The *Laplace-Runge-Lenz* vector defined as

$$\vec{A} = \vec{p} \times \vec{L} - m\,k\,\frac{\vec{r}}{r} \tag{1}$$

for a body under the influence of potential

$$V(r) = -\frac{k}{r} \tag{2}$$

is constant both in magnitude and direction.

Use these properties, together with angular momentum conservation, to find the equation that relates the radius and the angle describing the orbit, knowing that the magnitude of the LRL vector is

$$A = m\,k\,e\,, \tag{3}$$

where e is the eccentricity of the orbit.
Hint: consider $\vec{A} \cdot \vec{r}$ and use the invariance properties of the Levi-Civita-Symbol

Exercise 2. ***Perihelion precession***

Discuss the motion of a particle of reduced mass μ in the following central forces field,

$$F(r) = -\frac{k}{r^2} + \frac{C}{r^3} \tag{4}$$

This force is composed of the gravitation force and of a perturbing term proportional to the inverse cubic of the relative distance r. Assume that the additional force is very small compared to the gravitational force.

a) Show that the equation of the orbit can be cast in the following form

$$r = r(\phi) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\alpha\phi)} \tag{5}$$

where r, ϕ are the polar coordinates in the relative coordinate frame as discussed in the lecture. The parameters are $a = -k/2E$, where E is the energy of the orbit, while ε is the eccentricity of the orbit to be determined. The parameter α results from the presence of the perturbing term in the force and is responsible for a motion of *precession* (see next part). Find the values of ε and α in terms of the other constants of the problem.

Hint. Recall the form of the effective potential, which is valid for every central force.

The orbit (5) is closed and it is an ellipse when $\alpha = 1$, which correspondes to the Kepler problem (i.e. the case of $C = 0$). When $\alpha \neq 1$ the orbit is not closed and it is an ellipse with a motion of *precession*. The precession can be described by rotation of the *perihelion*, which is the inversion point of the orbit.

b) Write down the effective one-dimensional equation of motion for the central field (4) and convince yourself that it has the same form as the one for the Kepler problem, but with an augmented angular momentum. Relate the augmented part of the angular momentum into the precession of the elliptical orbit and show that the *angular speed* ω_p *of the precession* is

$$\omega_p = \frac{\pi}{T} \frac{1}{1-\varepsilon^2} \frac{C}{ka}, \tag{6}$$

where T is the period of the orbit.

Note that ω_p is expressed in terms of dimensionless parameter $\eta = C/ka$, which is the measure of the perturbation relative to the gravity.

c) The experimental observations have shown that the perihelion of Mercury precesses at a rate of $\omega_p/2\pi = 40$ arcseconds per century. Show that such a precession would require $\eta \approx 1.42 \cdot 10^{-7}$.

Hint. The eccentricity of the orbit of Mercury is $\varepsilon = 0.206$ and the period is $T = 0.24$ years.

Exercise 3. *Oscillations on a circle*

Consider three identical masses m, which are connected to each other by three identical springs. Denote the spring constant by k and let the rest length be L. In addition, the motion of the masses is restricted to a circle of radius R, as depicted in Fig. 1.

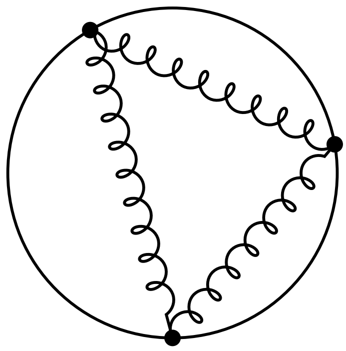

Figure 1: Three masses on sphere, connected by three springs.

a) Choose appropriate generalized coordinates and find the Lagrangian of the system.

b) Assume that $L = \frac{3}{2}R$ and find a minimum of the energy, in which non of the springs are completely contracted. Linearize the system in a neighborhood of it.

c) Find the eigensolutions of the linearized problem.

Exercise 4. ***Particle moving on a spiral***

Consider particle of mass m moving under the influence of an attractive central force of magnitude Cm/r^3.

a) Write down the second-order differential equation for r in terms of the effective potential. Show using $u = 1/r$ and the conservation of the angular momentum that it can be written as

$$u'' + u = \frac{C}{l^2} u$$

where $u' = \mathrm{d}u/\mathrm{d}\theta$ and l is angular momentum per unit mass. *Hint: look at problem 3, sheet 7.*

b) The particle is thrown at point A at distance a from the origin. Initially, it has speed v and moves in direction perpendicular to the line through point A and the origin. Show that if $v^2 < C/a^2$, the particle will spiral towards the origin and give the equation of its trajectory. Show further that it reaches the origin at time

$$T = \frac{a^2}{\sqrt{C - a^2 v^2}}.$$

Hint: you can assume the following identity: $\int_0^\infty \cosh^{-2} x = 1$.