Prologue

Given an orthonormal basis in a vector space with n dimensions, any vector can be represented by its components[1]

$$\vec{v} = \sum_{i=1}^{n} v_i \hat{e}_i. \tag{1}$$

In order to make formulae involving vectors less cumbersome, it is very useful to adopt the *Einstein summation convention*: repeated indices are implicitly summed over and the sign that indicates the sum omitted. For instance, we shall write

$$\vec{v} = v_i \hat{e}_i, \tag{2}$$

instead of the above formula. We will also be using extensively the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

As an example, the orthonormality condition reads

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}, \tag{4}$$

and a scalar product

$$\vec{v} \cdot \vec{w} = v_i \hat{e}_i \cdot w_j \hat{e}_j = v_i w_j \delta_{ij} = v_i w_i. \tag{5}$$

Two indices that are paired and summed over as in the last step on the right are sometimes said to be *contracted*.

Exercise 1. *The Levi-Civita symbol.*

Given a vector space of dimension n, the Levi-Civita symbol is an object with n indices defined by the property

$$\varepsilon_{\ldots i \ldots j \ldots} \equiv -\varepsilon_{\ldots j \ldots i \ldots}\,, \tag{6}$$

together with $\varepsilon_{12\ldots n} = +1$. We say that ε is *totally antisymmetric* under the exchange of any two indices.

(i) What is $\varepsilon_{i_1 \ldots i_n}$ equal to when two indices take the same value?
(ii) Assuming $s_{ij} = s_{ji}$, what can you say about $\varepsilon_{\ldots i \ldots j \ldots} s_{ij}$?
(iii) For $n = 2$, enumerate all values of the Levi-Civita symbol ε_{ij} and put them in a matrix.
(iv) For $n = 3$, list all non-zero values of the Levi-Civita symbol ε_{ijk}.

The practical examples of this course will mostly be set in euclidean space in three dimensions. Therefore we are going to work almost exclusively with ε_{ijk}, which will enable us to handle vector calculus in a very convenient way (see the next exercises). Given the following identity for the product of two Levi-Civita symbols

$$\varepsilon_{ijk}\varepsilon_{nlm} = \det \begin{pmatrix} \delta_{in} & \delta_{il} & \delta_{im} \\ \delta_{jn} & \delta_{jl} & \delta_{jm} \\ \delta_{kn} & \delta_{kl} & \delta_{km} \end{pmatrix}; \tag{7}$$

[1] In differential geometry, it is important to distinguish between upper and lower indices. For this course such distinction is not required (if you are wondering why, the reason is that we will only deal with euclidian spaces).

(v) Show that $\varepsilon_{ijk}\varepsilon_{ilm} = \det\begin{pmatrix}\delta_{jl} & \delta_{jm}\\ \delta_{kl} & \delta_{km}\end{pmatrix} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

(vi) Show that $\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}$.

(vii) Show that $\varepsilon_{ijk}\varepsilon_{ijk} = 6$.

One of the possible definitions of the vector product reads

$$\vec{v} \times \vec{w} \equiv \varepsilon_{ijk} v_j w_k \hat{e}_i. \tag{8}$$

(viii) Show that, also according to this definition, $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, and that its length is equal to the area spanned by a parallelogram with sides $\vec{v}$ and $\vec{w}$. Can you say to which property of ε the right-hand rule is related?

Exercise 2. ***Vector Identities***

Prove the following identities:

1. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$
2. $|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\,|^2 = (\mathbf{a} \cdot \mathbf{c})^2\,\mathbf{b}^2 + (\mathbf{a} \cdot \mathbf{b})^2\,\mathbf{c}^2 - 2\,(\mathbf{a} \cdot \mathbf{c})\,(\mathbf{a} \cdot \mathbf{b})\,(\mathbf{b} \cdot \mathbf{c})$
3. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$
4. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c})$
5. $\mathbf{Ra} \times \mathbf{Rb} = \mathbf{R}\,(\mathbf{a} \times \mathbf{b})$
6. $\nabla \times \nabla\psi = 0$
7. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
8. $\nabla \times (\nabla \times \mathbf{A}) = \nabla\,(\nabla \cdot \mathbf{A}) - \Delta\mathbf{A}$
9. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
10. $\nabla\,(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\,\mathbf{B} + (\mathbf{B} \cdot \nabla)\,\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$
11. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B}$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are vectors, $\mathbf{A}$, $\mathbf{B}$ are vector fields, ψ is a function and $\mathbf{R} \in \mathrm{SO}(3)$. Moreover assume that all components A_i, B_j and also ψ are in $\mathcal{C}(2)$, i.e. two times continuously differentiable.
Don't write out cross products explicitly, but use the index notation involving the Levi-Civita symbol ε_{ijk}.

Exercise 3. ***Jacobian***

In this exercise, we will have a closer look at the Jacobian for the example of the polar coordinate transformation.

a) Calculate the Jacobian matrix $J(r, \Theta)$ for the polar coordinate transformation $x = r\cos\Theta$, $y = r\sin\Theta$.

b) Show that $dx = \cos\Theta dr - r\sin\Theta d\Theta$, $dy = \sin\Theta dr + r\cos\Theta d\Theta$ and calculate the area element $dx \times dy$ in terms of dr and $d\Theta$.

c) In the expression for $dx \times dy$, where does the Jacobian come in?

d) When does the determinant of $J(r, \Theta)$ vanish? The inverse function theorem states that if a continuously differentiable function has a non-zero Jacobian determinant at some point, the function is invertible in an open region containing that point. What does this theorem tell you about the polar coordinate transformation we looked at in this exercise?

Exercise 4. ***Vector Calculus***

This exercise requires using various techniques that you have learnt in this sheet.

Calculate $\partial r/\partial x_i$ where $r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$. A vector field $\mathbf{u}$ is given by

$$\mathbf{u} = \frac{\mathbf{a}}{r} + \frac{(\mathbf{a} \cdot \mathbf{x})\mathbf{x}}{r^3}.$$

Find the Jacobian $J_{ij} = \partial u_i/\partial x_j$ and deduce that $\nabla \cdot \mathbf{u} = 0$.