Ginzburg criterion for mean field theories:

For a general GL functional, we consider fluctuations in the order parameter close to T_c. Here $|\psi| \ll 1$ and quartic terms can be neglected.

$$F = \int d^d x \left[\alpha(T) |\psi|^2 + K |\nabla \psi|^2 \right] \qquad K = \frac{1}{2m}$$

Using $\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k \tilde{\psi}(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$

$$F = \sum_k \left[\alpha(T) + K |\vec{k}|^2 \right] |\tilde{\psi}(\vec{k})|^2.$$

The Helmholtz free energy,

$$e^{-\frac{\mathcal{F}}{k_B T}} = \int D[\hat{\psi}, \hat{\psi}^*] \, e^{-\frac{F}{k_B T}}.$$

$D[\psi, \psi^*] \rightarrow$ functional integral measure.

Converting to the fourier space variables,

$$D[\tilde{\psi}, \tilde{\psi}^*] = \prod_{\vec{q}} d\tilde{\psi}(\vec{q}) \, d\tilde{\psi}(-\vec{q}).$$

$$\equiv \prod_{\vec{q}} d\,\mathrm{Re}\,\tilde{\psi}(\vec{q}) \, d\,\mathrm{Im}\,\tilde{\psi}(\vec{q})$$

For each $\vec{q}$ mode, the functional integral leads to a normal Gaussian integral.

The full functional integral is hence a product of all the integrals for the $\vec{q}$ modes.

$$e^{-\frac{\mathcal{F}(T)}{k_B T}} \equiv Z = \prod_{\vec{k}} \left[\frac{\pi k_B T}{\alpha(T) + K|\vec{k}|^2} \right]$$

The resulting free energy arising from fluctuations is

$$f(T) = -k_B T \ln Z = -k_B T \sum_{\vec{k}} \ln \frac{\pi k_B T}{\alpha + K|\vec{k}|^2}.$$

Specific heat $C_V = -T\mathcal{F}''(T)$

$$= k_B T^2 \sum_{\vec{k}} \frac{[\alpha'(T)]^2}{[\alpha(T) + K|\vec{k}|^2]^2} + \text{other terms} \quad (1)$$

Since we are interested in behaviour in the critical region, we only retain the most singular contributions in (1):

Now Consider specific heat / unit volume $\Rightarrow$

$C \equiv \frac{C_V}{k_B N}$ $\longrightarrow$ N # of lattice sites

$N = \frac{V}{a^d}$ $\quad a \rightarrow$ lattice spacing.

$$C = T^2 a^d \int \frac{d^d k}{(2\pi)^d} \cdot \frac{[\alpha'(T)]^2}{[\alpha(T) + K|\vec{k}|^2]^2} + \cdots$$

* Note: While differentiating $f(T)$ for $C_V(T)$, you might encounter terms like $\sum_{\vec{k}} (\text{const!})$ in C_V. These are not really divergent as the integrals have a cut-off. Moreover, various other contributions at finite T have been neglected which would impact these "constant large terms".

Physics of fluctuations qualitatively similar on both sides of T_c.

In general

$$f = f_0 + \int_V \underbrace{\alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4}_{f_{mf}} + K(\nabla \Psi)^2 \quad \text{— (a)}$$

when $T > T_c$ $[T = T_c^+]$ then $|\psi_0|^2 = 0$ [obtained by minimising f_{mf}]

then fluctuation free energy is using $\Psi = \psi_0 + \psi$,

$$F \approx \int_V \alpha |\psi|^2 + K(\nabla \psi)^2 \quad \text{— (b)}$$

When $T < T_c$ then $|\psi_0|^2 = -\frac{\alpha}{\beta}$.

To obtain fluctuation free energy for $T < T_c$, need to subtract mean field contribution, then using $\Psi = \psi_0 + \psi$ in (a),

$$F \approx \int_V -2\alpha|\psi|^2 + K(\nabla\psi)^2 = \int_V \left[2|\alpha| \, |\psi|^2 + K(\nabla\psi)^2 \right] \quad \text{— (c)}$$

So the coefficients α and $2|\alpha|$ in both (b) and (c) are +ve for any T. Fluctuations are important on both sides.

∴ for $T > T_c$ we have ξ^{-2} in the integrand for C
& $T < T_c$ " " $2\xi^{-2}$ " " "

This illustrates that fluctuation exponents are the same on either side of T_c but there are quantitative differences.

$$a(T) = \frac{\bar{\alpha}(T-T_c)}{T_c} . \qquad a'(T) = \frac{\bar{\alpha}}{T_c}$$

also, considering $T \gtrsim T_c$. $\quad a(T) > 0$.

$$C_{sing} = \frac{\bar{\alpha}^2 a^d}{K^2} \frac{T^2}{T_c^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{[\xi^{-2} + |\vec{k}|^2]^2} .$$

$$\xi = \sqrt{\frac{K}{|\alpha|}} \; \bar{\alpha} \; |t|^{-1/2} \quad \text{where } t = \frac{T-T_c}{T_c} .$$

↓ coherence length in superconductors.

$$C(t)_{sing} \approx \frac{\bar{\alpha}^2 a^d}{K^2} \xi^4 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(1+\xi^2|\vec{k}|^2)^2} .$$

Changing variables, $\vec{q} = \xi \vec{k}$

$$C(t) \approx \frac{\bar{\alpha}^2 a^d \xi^{4-d}}{K^2} \int_0^{\Lambda\xi} \frac{q^{d-1}\, dq\, d\Omega}{(2\pi)^d} \frac{1}{(1+q^2)^2} .$$

Cut-off $\Lambda \sim \frac{1}{a}$ in real systems. [so no real divergence of integral]

Leading "t" dependence

$$C(t) \sim \begin{cases} \cancel{\text{const}}\; a^{4-d} & d > 4 \\ -\ln|t| & d = 4 \\ |t|^{\frac{d}{2}-2} & d < 4 . \end{cases}$$

[Integral is finite]

Mean-field is qualitatively accurate in $d > 4$ with finite corrections

Fluctuation contributions are divergent in other dimensions < 4 as $|t| \to 0$.

Mean field is ok if , $C_{sing} < \Delta C$ (jump in sp. heat)

$$\frac{\bar{\alpha}^2}{K^2} a^d \xi^{4-d} \ll 1$$

Using $\xi = \left[\frac{K}{\bar{\alpha}|t|}\right]^{1/2}$.

we can define a temperature scale, t_G.

such that $$\frac{\alpha^2 a^d}{K^2} \left[\frac{K}{\bar{\alpha}}\right]^{2-\frac{d}{2}} \cdot |t_G|^{\frac{d}{2}-2} = 1$$

$$\Rightarrow \quad |t_G| = \left[\frac{a}{R}\right]^{\frac{2d}{4-d}} \quad \text{where} \quad R = \sqrt{\frac{K}{\bar{\alpha}}}.$$

A similar scale is obtained starting from the ordered side too.

Mean field ok if $|t| \gg \{t_G\}$.

For lattice ferromagnets in 3D $R \sim a \Rightarrow t_G \sim 1$.
critical regime where mean field does not work is very large.

In conventional low-T SC. (s-wave).

$R \sim$ Cooper pair size $\sim (10^2 - 10^3)\, a$.

$\Rightarrow t_G \sim 10^{-18} \sim 10^{-12} \Rightarrow$ critical regime negligeably narrow and Mean field works!

In unconventional superconductors, like high T_c, Cooper pairs are confined to smaller sizes and hence ξ decreases. For some materials like $YBa_2Cu_3O_{7-y}$ (the famous YBCO) with $T_c \sim 100\,K$ $\xi \sim 10\,\mathring{A} \Rightarrow t_G \sim 10^{-2}$ or 10^{-1} which implies there must be sizable deviations from such mean field predictions.