A Short Introduction to Topological Superconductors

--- A Glimpse of Topological Phases of Matter

Jyong-Hao Chen
陳炯豪

Condensed Matter Theory, PSI & Institute for Theoretical Physics, ETHZ

PAUL SCHERRER INSTITUT PSI

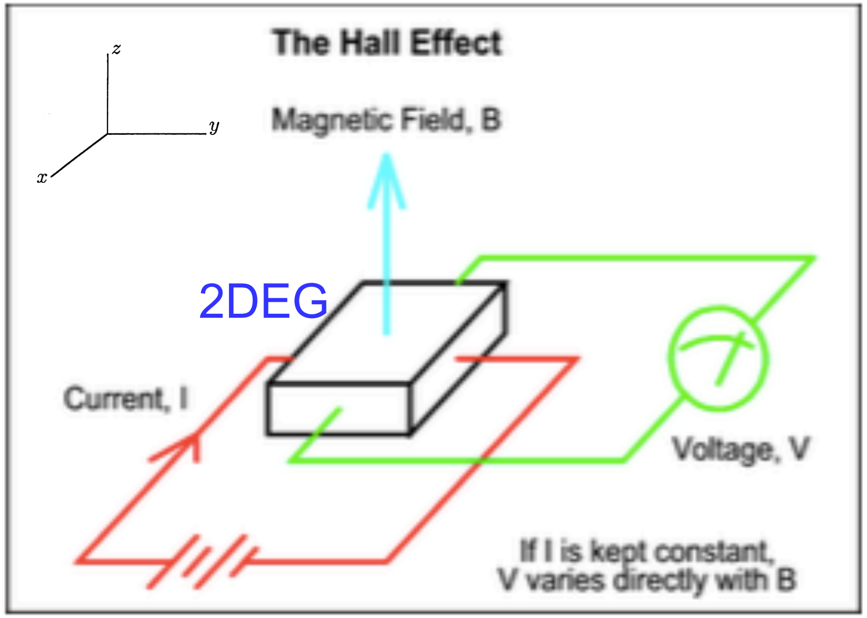
ETH Zürich

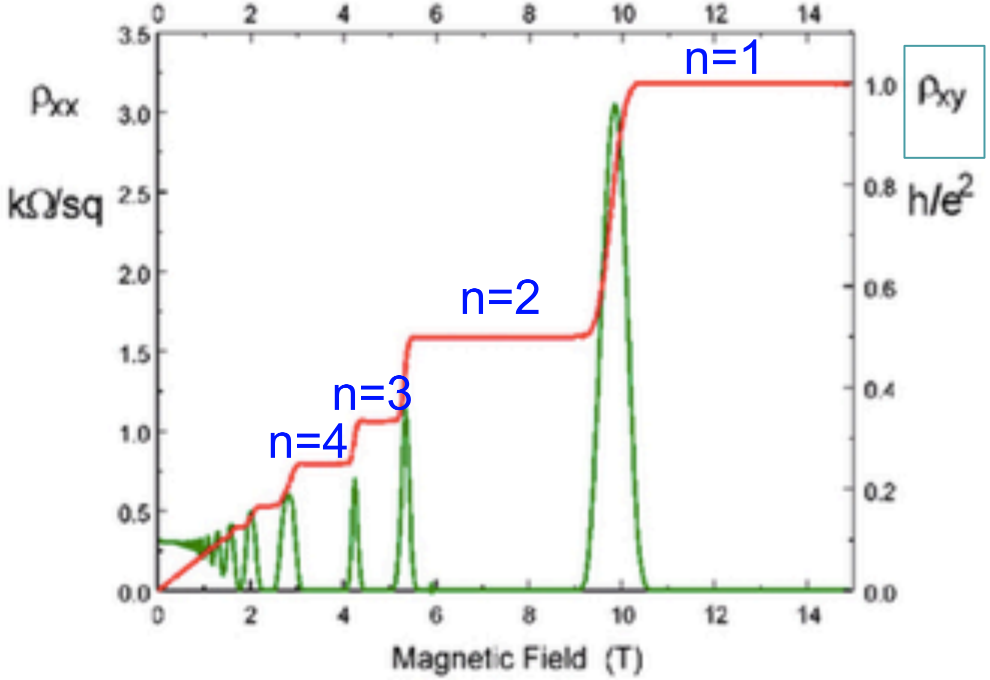
Dec. 09, 2015 @ Superconductivity Course, ETH

jyong-hao.chen@psi.ch

I. Introduction

1980: Integer quantum Hall effect





$$\rho_{xy} = \frac{V}{I}$$

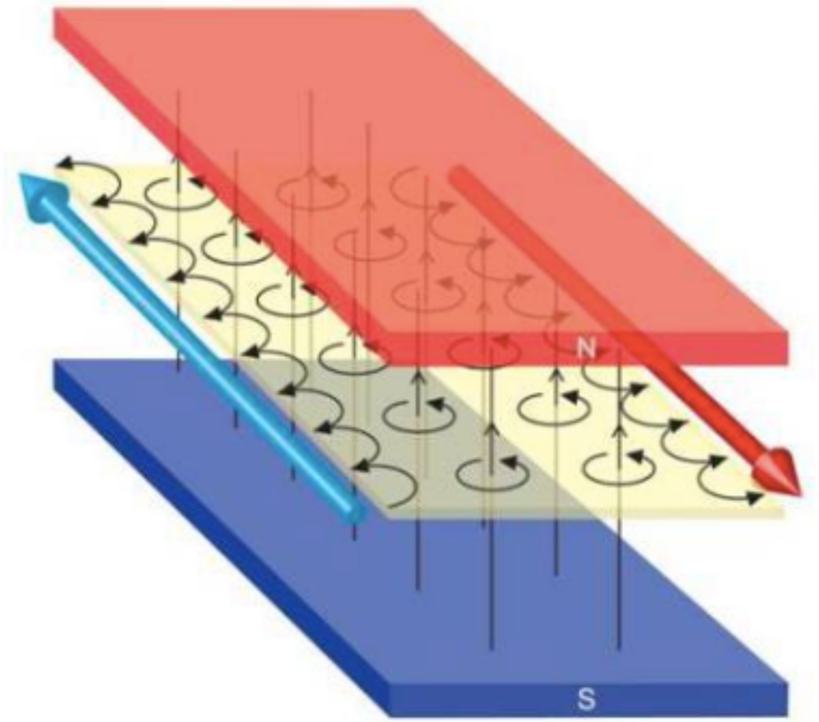
- Quantized Hall resistance $\rho_{xy} = \frac{1}{n}\frac{h}{e^2} \times (1 \pm 10^{-9})$

Nobel Prize 1985: von Klitzing

Quantum Hall Effect: chiral gapless edge states

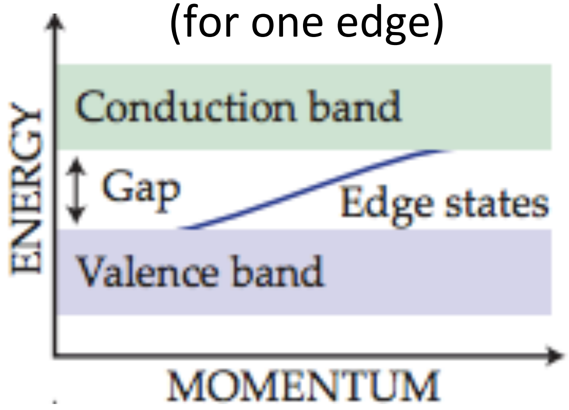
Halperin, 1982

Chiral edge state
(skipping orbit picture)



Gapless excitations at the edges

Energy spectrum
(for one edge)



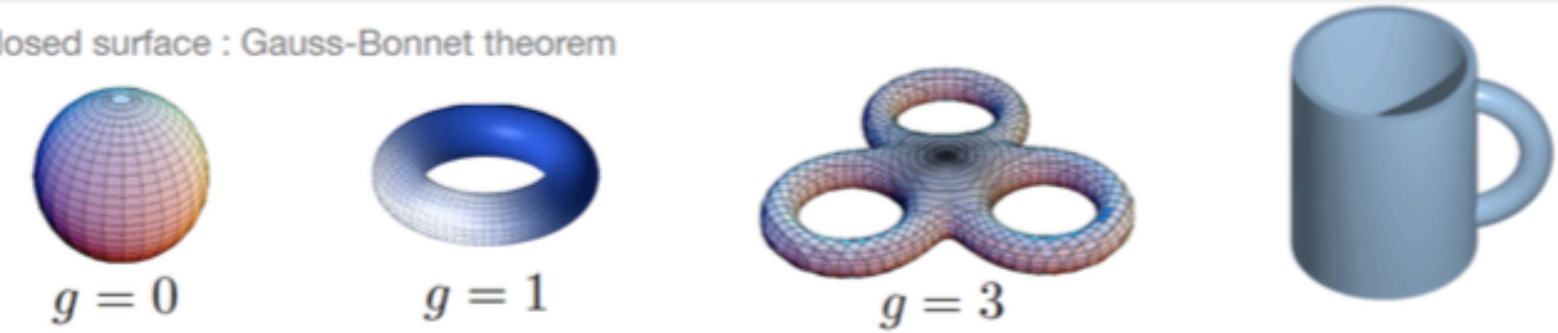
✧ **Robust** chiral gapless edge states against disorder
(no back scattering)

1982: Rise of Topology

- Quantized Hall conductance $\rho_{xy} = \frac{1}{n}\frac{h}{e^2}$ has a topological origin

- Topological properties – insensitive to small changes of parameters of the manifold (here, band structure)

For closed surface : Gauss-Bonnet theorem



$$\int dS\ \kappa = 2\pi(2-2g)$$

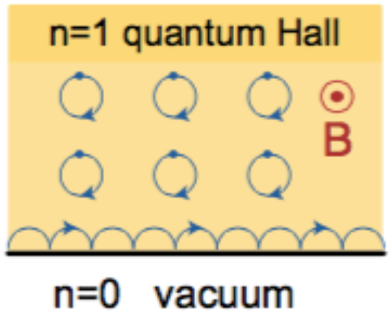
- Integral of curvature depends only on topology, insensitive to small deformation of the manifold ➔ topological invariant

Thouless, Kohmoto, Nightingale, and den Nijs, 1982

- TKNN invariant = the first Chern number = integer

$$n = \int \frac{d^2k}{(2\pi)^2}\varepsilon^{\mu\nu}F_{\mu\nu}(k) \qquad \mathcal{A}_i^{(m)}(\vec{k}) = i\langle u_m(\vec{k})|\nabla_{k_i}|u_m(\vec{k})\rangle$$

Berry curvature   Berry connection



- Universal manifestation: gapless excitation at interface between topologically distinct regions

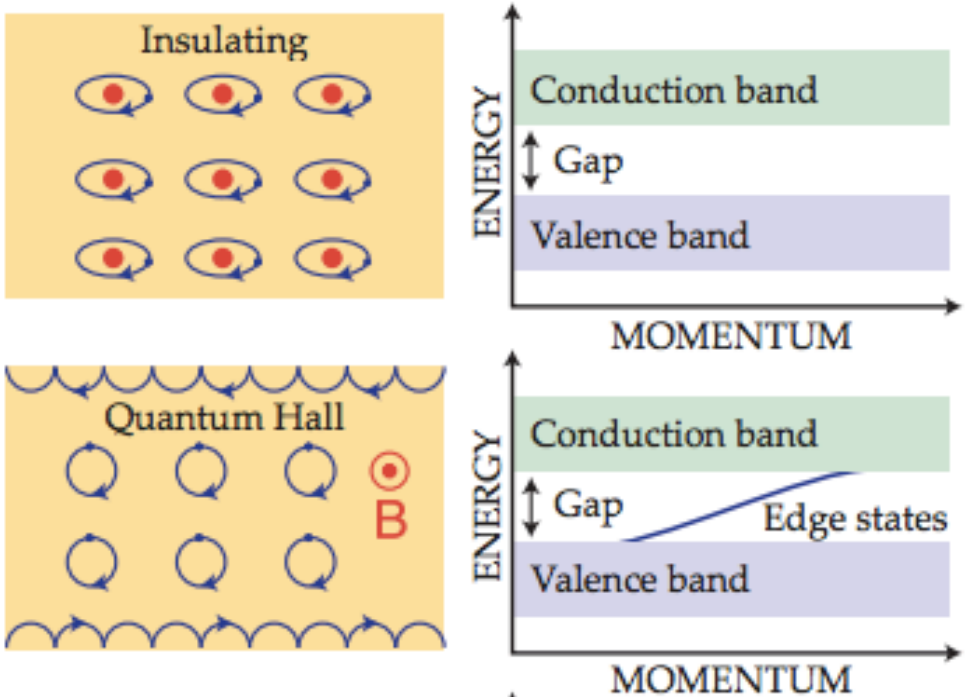
General characterization for *topological phases of matter*

- Bulk excitation is gapped, possess a bulk topological invariant
- Protected gapless excitations at the boundary

Bulk-boundary correspondence

trivial insulator
v.s.
quantum Hall

Robust chiral edge state, as long as band topology does not change



➢ Can we have more examples of topological phases of matter?

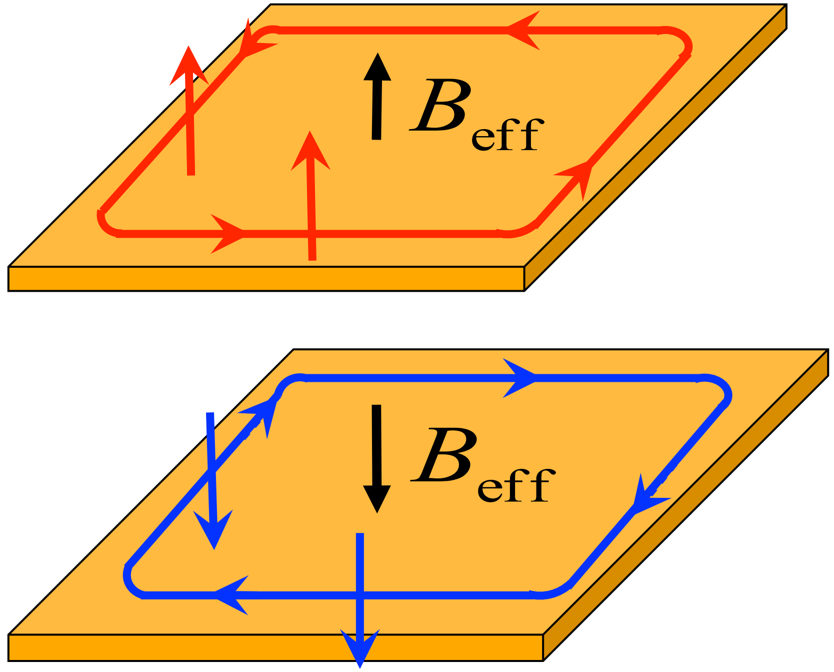
✧ Presence of time-reversal symmetry gives rise to new topological phases

Kane-Mele, PRL 05

2005: Quantum Spin Hall Effect/Z_2 topological insulator

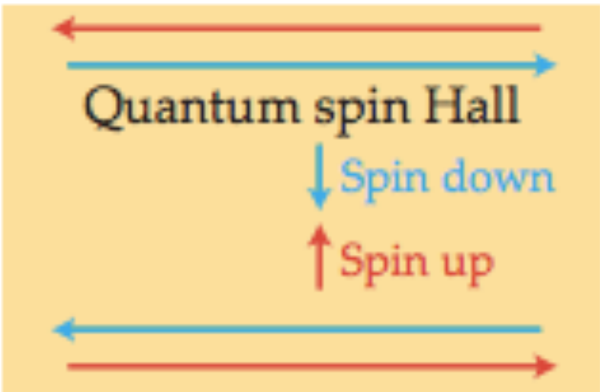
- QSH = two copies of QH states, one for each spin component, each seeing the opposite magnetic field.
- Time reversal symmetric, and can exist without any external magnetic field.
- Effective magnetic field: Spin-orbital coupling

$$H_{so} = \lambda_{so}\vec{\sigma}(\vec{p}\times\vec{E})$$

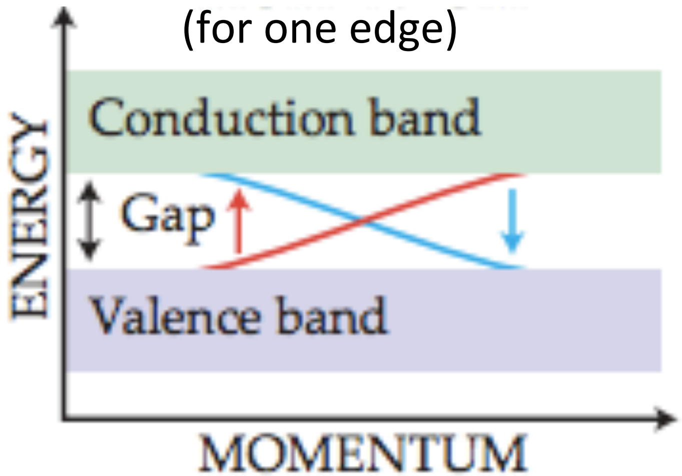


Bernevig and Zhang, 2006

helical edge states

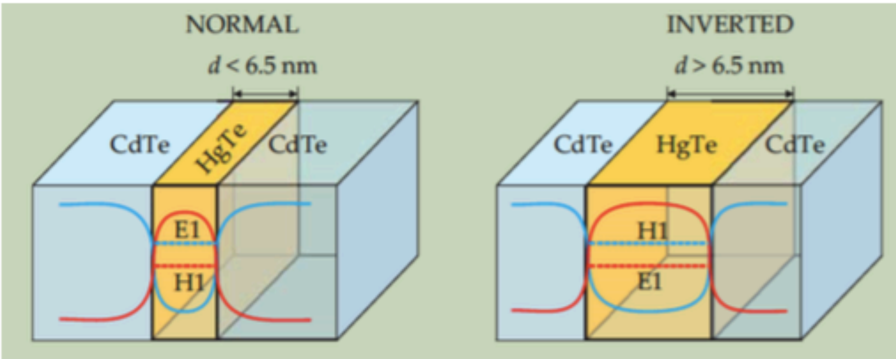


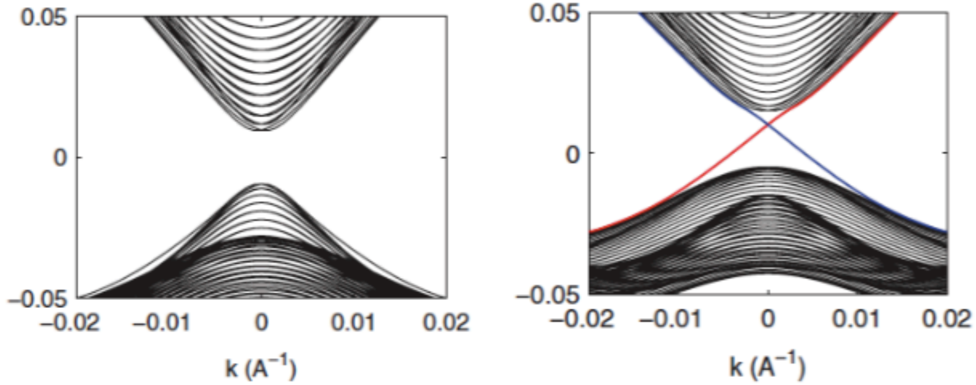
Energy spectrum
(for one edge)



Experimental observation of HgTe TI

- **Theoretically predicted in 2006** Bernevig, Hughes, and Zhang, Science, 2006

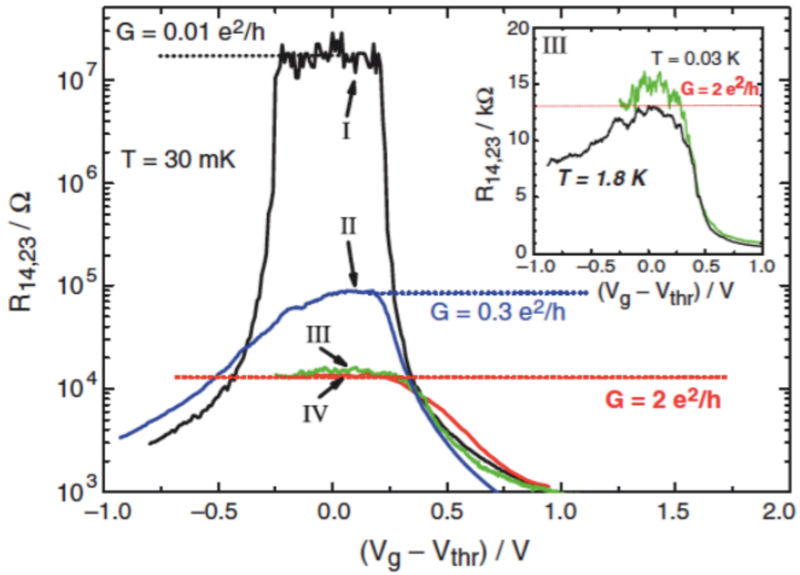


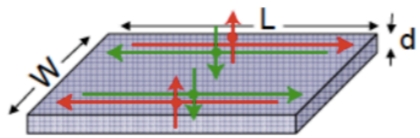


- **Experimentally found in Nov. 2007** Konig et al., Science, 2007

– Measure conductance while tuning E_F through the bulk energy gap

– edge state conductance $2e^2/h$ observed independent of W and L





I. d=5.5nm (normal) Insulating in gap

II-IV d=7.3nm (inverted) conducting in gap

II. L = 20 μm (> L_{in})

III. L = 1 μm W = 1 μm

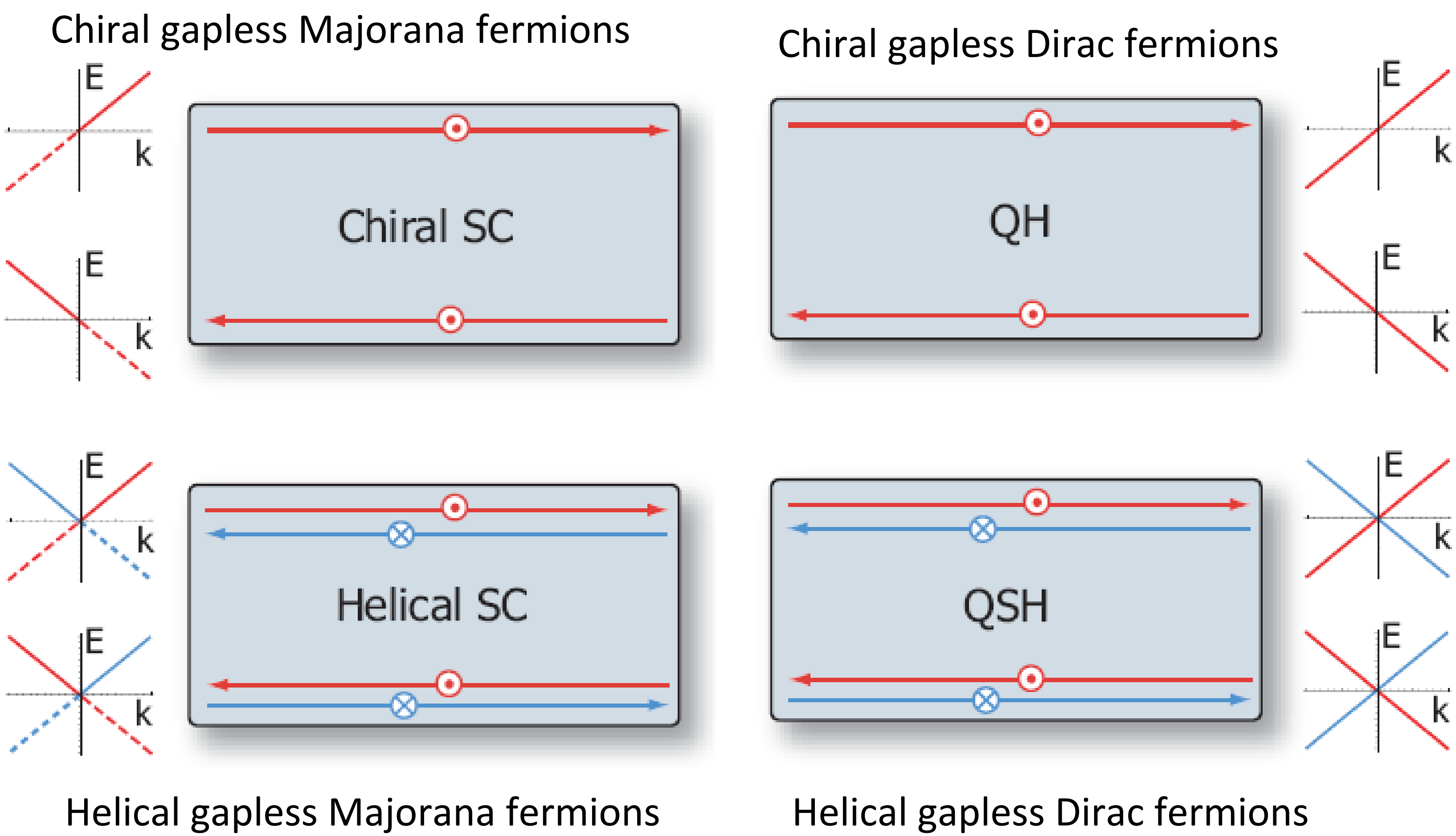
IV. L = 1 μm W = .5 μm

➢ **Nobel Prize in 20XX?**

New topological phases of matter

chiral superconductor,
helical superconductor

quantum Hall insulator,
quantum spin Hall insulator



Majorana fermion:
particle = antiparticle

✧ Quasiparticle excitations in superconductors posses all the key attributes of Majorana fermions

II. TSC

BCS, BdG, and particle-hole symmetry

- BCS mean field theory: $c^\dagger c\, c^\dagger c \quad \Rightarrow \quad \langle c^\dagger c^\dagger \rangle c\, c = \Delta^* c\, c$

$$H = \frac{1}{2}\sum_{\boldsymbol{k}} \begin{pmatrix} c^\dagger & c \end{pmatrix} H_{\mathrm{BdG}} \begin{pmatrix} c \\ c^\dagger \end{pmatrix}$$

Bogoliubov-de Gennes Hamiltonian

$$H_{\mathrm{BdG}} = \begin{pmatrix} h_0 & \Delta \\ \Delta^\dagger & -h_0^T \end{pmatrix}$$

- Built-in anti-unitary particle-hole symmetry

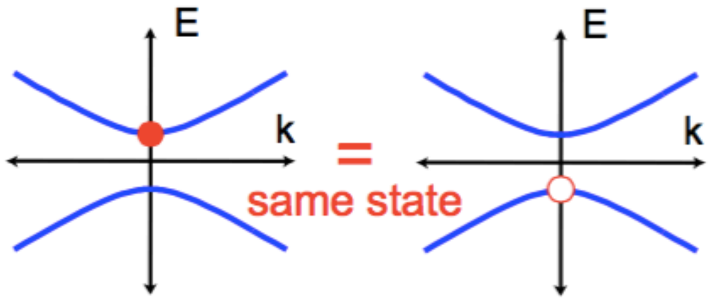
$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boxed{\Xi\, \mathcal{H}_{\mathrm{BdG}}(\boldsymbol{k})\, \Xi^{-1} = -\mathcal{H}_{\mathrm{BdG}}(-\boldsymbol{k})} \qquad \Xi = \tau_x \mathcal{K} \qquad \Xi\varphi = \tau_x \varphi^*$$

- Bogoliubov quasiparticle

$$\varphi_{-E} = \Xi\varphi_E \qquad (u^*_{-k} = v_k)$$

$$\Rightarrow \quad \gamma_E^\dagger = \gamma_{-E}$$ Majorana condition



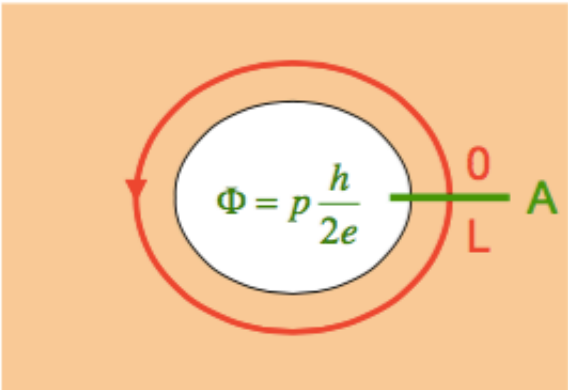
Majorana fermion:
particle = antiparticle

- Zero energy solution E = 0: Majonana zero mode

Majorana zero mode at a vortex

$$\Delta = \Delta(r) e^{i\varphi}$$ in $P_x + iP_y$ superconductor

Volovik 1999;
N. Read, D. Green, 2000

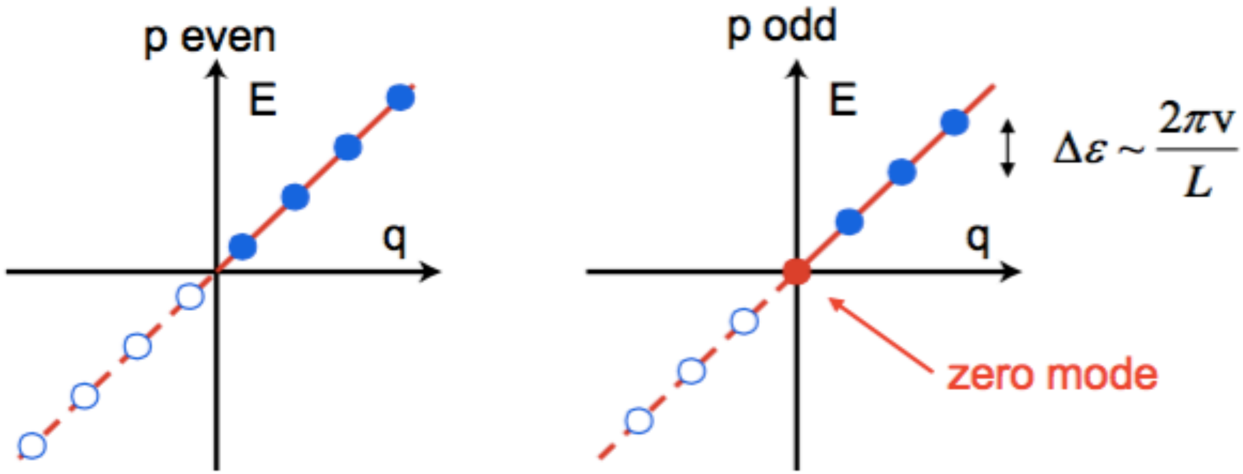


Hole in a topological superconductor threaded by flux

Boundary condition on fermion wavefunction

$$\psi(L) = (-1)^{p+1}\psi(0)$$

$$\psi(x) \propto e^{iq_m x} \quad ; \quad q_m = \frac{\pi}{L}(2m+1+p)$$



Ivanov, 2001

- Majorana zero modes obey non-Abelian statistics: non-Abelian anyons
- Application to quantum computation: topological quantum computation

Potential material candidate for Majorana zero modes

- The only currently realized bulk (P+iP) superconductors are

  - superfluid ^{3}He-A
  - unconventional superconductors: Sr_2RuO_4

➢ Do Majoranas occur elsewhere?

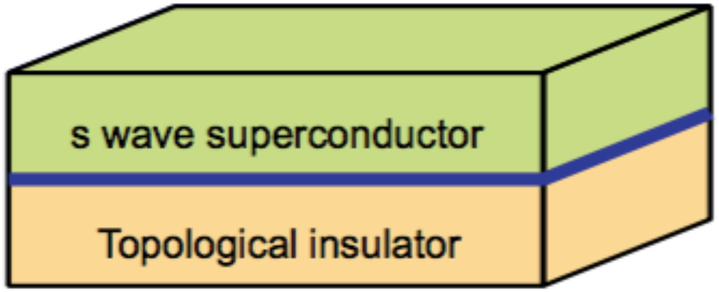
Superconducting proximity effect

Fu & Kane, PRL, 08

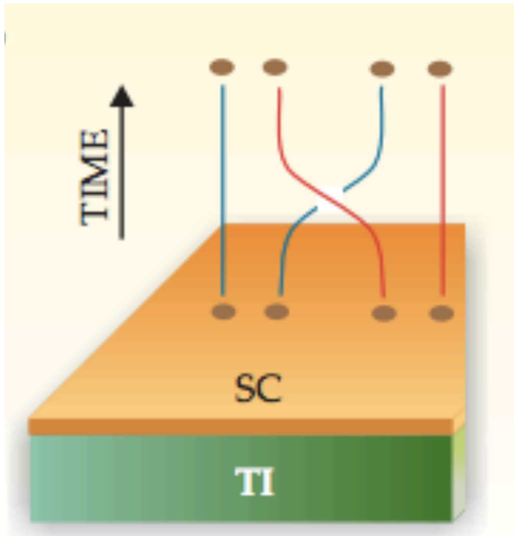
Minimal surface state model:

$$H_0 = \psi^\dagger(-iv\vec{\sigma}\cdot\vec{\nabla} - \mu)\psi$$

$$V_S = \Delta\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*\psi_\downarrow\psi_\uparrow$$

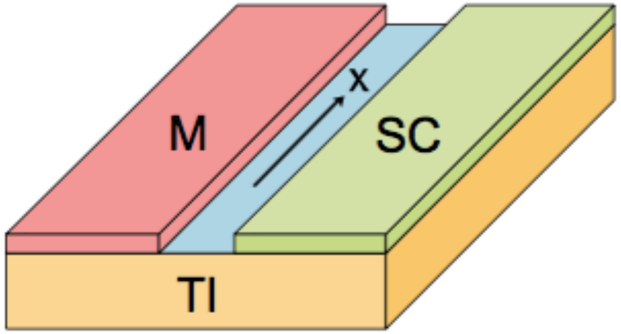


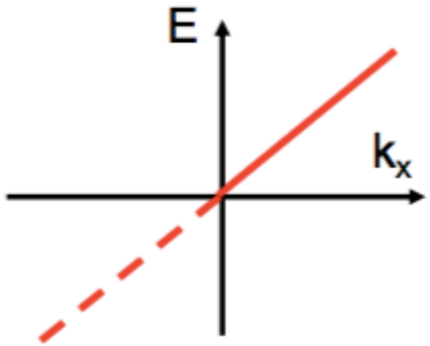
- Nontrivial ground state supports Majorana zero mode at vortices



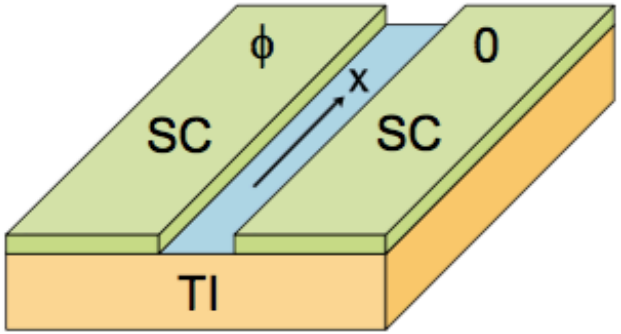
More examples:
mimic 1D Majorana edge states on TI

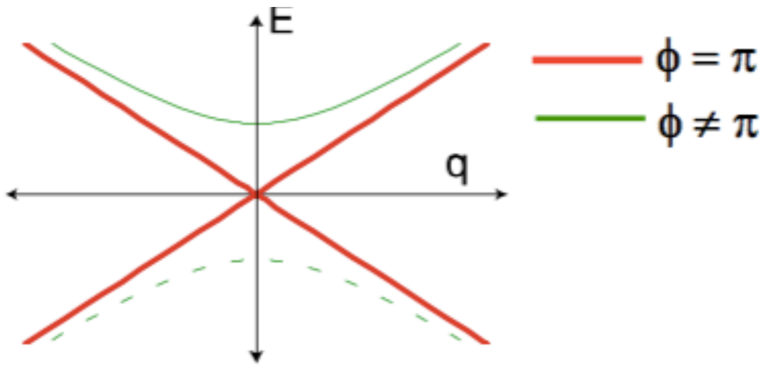
- 1D chiral Majorana edge states at superconductor-magnet interfaces





- 1D helical Majorana edge states at SC-TI-SC Josephson junction





$$H = -i\hbar v_F \left(\gamma_L \partial_x \gamma_L - \gamma_R \partial_x \gamma_R\right) + i\Delta\cos(\phi/2)\gamma_L\gamma_R$$

III. Classification

Altland-Zirnbauer's ten-fold way

10 = 3x3 +1

- Quadratic Hamiltonian $H = \sum_{AB} \psi_A^\dagger \mathcal{H}_{AB} \psi_B$
- Symmetries

A. Altland and M. R. Zirnbauer, 1997

Anti-Unitary Symmetries :

- Time Reversal : $\Theta H(\mathbf{k}) \Theta^{-1} = +H(-\mathbf{k})\ ; \quad \Theta^2 = \pm 1$

- Particle - Hole : $\Xi H(\mathbf{k}) \Xi^{-1} = -H(-\mathbf{k})\ ; \quad \Xi^2 = \pm 1$

Unitary (chiral) symmetry : $\Pi H(\mathbf{k}) \Pi^{-1} = -H(\mathbf{k})\ ; \quad \Pi \propto \Theta\Xi$

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Kitaev, 2008;
Schnyder, Ryu, Furusaki, Ludwig, 2008

Bott Periodicity d→d+8

Interaction effect

- For example, in some case, breakdown of the topological classification *Z*

T. Morimoto, A. Furusaki, C. Mudry, 2015

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	+1	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	R_{1-d}	$\mathbb{Z}_8, \mathbb{Z}_4$	0	0	0	$\mathbb{Z}_{16}, \mathbb{Z}_8$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	+1	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_{32}$	0
AII	−1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	R_{5-d}	$\mathbb{Z}_2, \mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}, \mathbb{Z}_{16}$	0	0	0
C	0	−1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	+1	−1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

Learn more?

✧ Review articles

- D. Xiao, M.C. Chang, and Qian Niu, Rev. Mod. Phys. **82**, 1959, (2010) — Berry phase

- M.Z. Hasan and C.L. Kane, Rev. Mod. Phys. **82**, 3045 (2010)
- X.L. Qi and S.C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011)

TI

- J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012)
- S.R. Elliott and M. Franz, Rev. Mod. Phys. **87**, 137 (2015)
- C.K.J. Beenakker, Rev. Mod. Phys. **87**, 1037 (2015)

TSC

- C.-K. Chiu, J.C.Y. Teo, A.P. Schnyder, S. Ryu, arXiv:1505.03535 — Classification

- The Net Advance of MIT http://web.mit.edu/redingtn/www/netadv/

Alphabetical Index to Review Papers and Tutorials:
A -- B -- C -- D -- E -- F -- G -- H -- I -- J -- K -- L -- M -- N -- O -- P -- Q -- R -- S -- T -- U -- V -- W -- X -- Y -- Z

The Net Advance of Physics:
"T"

T Tauri Stars --- Tachyons --- Tau Leptons --- Tau Neutrinos --- Technicolor --- Teleportation (Quantum) --- Telescopes --- Temperature --- Textures (High Energy Physics) --- Thermal Field Theory --- Thermistors --- Thermodynamics --- Thermo-Electric Materials --- Thin Films --- Tides --- Time --- Time-Keeping --- Tin Alloys --- t-j Model --- Tomography --- Tomonaga-Luttinger Liquids --- Top Quark --- Topological Field Theory --- Topological Insulators --- Topological String Theory --- Topology --- Topology of the Universe --- Toric Varieties --- Torsion Tensor --- Torus --- Traffic Flow --- Transactional Interpretation --- Transients --- Transport Theory --- Transversivity --- Triaxial Potential --- Tropical Mathematics --- Tsunamis --- Turbulence --- Turing Machines --- Tweed --- Twistors --- Two-Body Problem

✧ Youtube

- Charles Kane, Sou-Cheng Zhang, A.J. Leggett, …

- Prospects in Theoretical Physics 2015-Princeton Summer School on Condensed Matter Physics

https://pitp.ias.edu/2015/
(Charles Kane, Edward Witten, Xiao-Gang Wen…)

arXiv.org > cond-mat > arXiv:1510.07698

Condensed Matter > Mesoscale and Nanoscale Physics

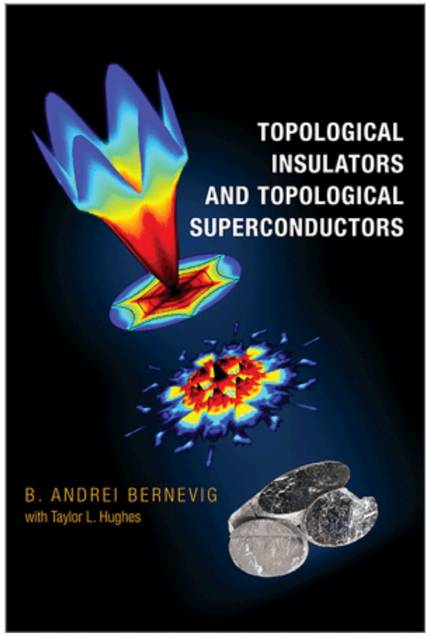
Three Lectures On Topological Phases Of Matter

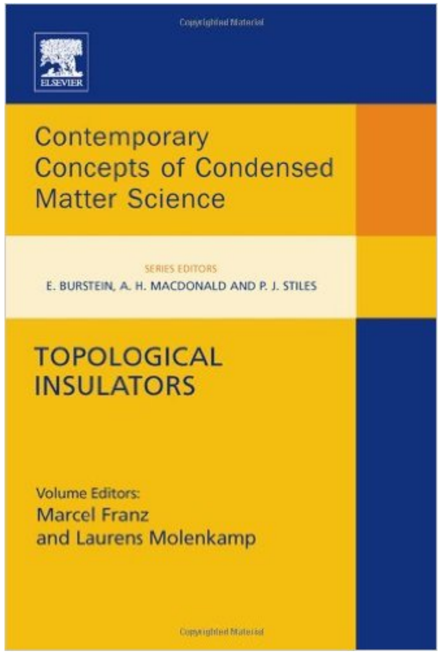
Edward Witten

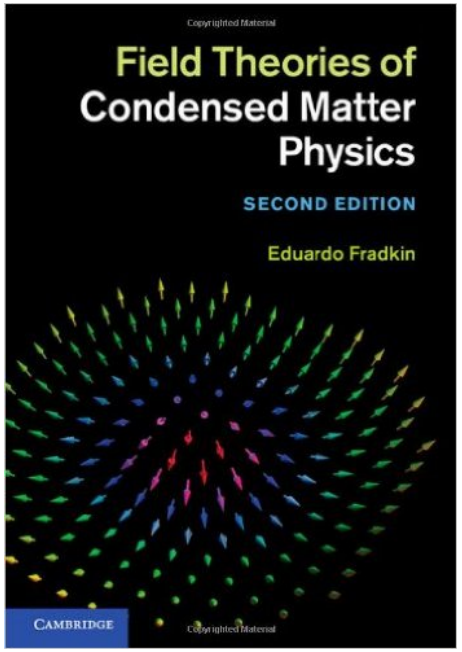
(Submitted on 26 Oct 2015)

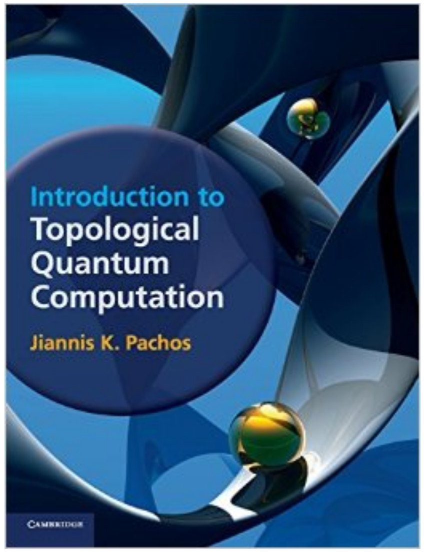
These notes are based on lectures at the PSSCMP/PiTP summer school that was held at Princeton University and the Institute for Advanced Study in July, 2015. They are devoted largely to topological phases of matter that can be understood in terms of free fermions and band theory. They also contain an introduction to the fractional quantum Hall effect from the point of view of effective field theory.

✧ **Books**


TOPOLOGICAL
INSULATORS
AND TOPOLOGICAL
SUPERCONDUCTORS
B. ANDREI BERNEVIG
with Taylor L. Hughes


Contemporary Concepts of Condensed Matter Science
SERIES EDITORS
E. BURSTEIN, A. H. MACDONALD AND P. J. STILES
TOPOLOGICAL INSULATORS
Volume Editors:
Marcel Franz
and Laurens Molenkamp


Field Theories of
Condensed Matter
Physics
SECOND EDITION
Eduardo Fradkin
CAMBRIDGE


Introduction to
Topological
Quantum
Computation
Jiannis K. Pachos