Binding and dimensionality:

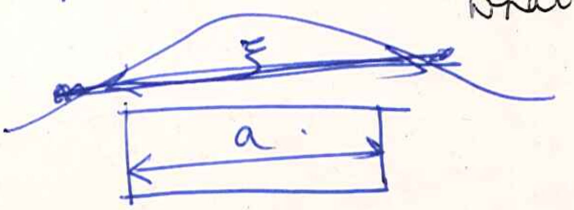
Consider a spherically symm pot. $U(r) = -U_0\,\Theta(a-r)$.

Are there bound states?

Clearly yes if U_0 is large!

Consider a trial state localized with ξ inside potential well. If K.E of state $> U_0$ state delocalizes.

What happens when $U_0 < T_0$? $\lambda > 1$.



$$\xi = \lambda a$$

Variational energy $\rightarrow$ $$E \simeq \frac{\hbar^2}{2m\xi^2} - U_0\left(\frac{a}{\xi}\right)^d.$$

$$= \underbrace{\frac{\hbar^2}{2ma^2}}_{T_0} \cdot \lambda^{-2} - U_0\lambda^{-d}$$

diff. w.r.t. λ

$$-2T_0\,\lambda^{-3} + d\,U_0\lambda^{-(d+1)} = 0$$

$$\Rightarrow \quad \lambda = \left(\frac{2T_0}{dU_0}\right)^{\frac{1}{2-d}}$$

$$\therefore\ E = -\left(\frac{2}{d}\right)^{2/(d-2)}\left(1-\frac{2}{d}\right) T_0^{\,d/(d-2)}\, U_0^{\,-2/(d-2)}$$

$d=1$, $\quad E < 0 = -\dfrac{U_0^2}{4T_0}$ $\qquad \lambda = \dfrac{2T_0}{dU}$.

$d=2$, $\cancel{\lambda \rightarrow}$ & $E \simeq 0$. (localized states with ballistic dip..

$d>2$ $\quad E>0$ $\quad$ no bound states "

Instability of Fermi surface to attractive x^{ns}.

What is a Fermi surface?

Simple model of 2 e^{-s}. $\vec{r_1}$ + $\vec{r_2}$ (other e^{-s} are like free particles)

Assume the 2 chosen e^{-s} do not x^{ct} with others

Other e^{-s} forbid them from occupying levels with $k < k_F$.

Behaviour of relative coord: $(\vec{r_1} - \vec{r_2})$

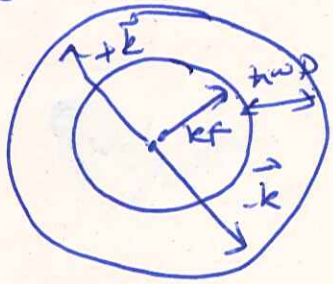
$$\psi(\vec{r_1}, \vec{r_2}) = \psi(\vec{r_1} - \vec{r_2}) = \sum_k g(\bar{k}) e^{i\bar{k}\cdot(\vec{r_1} - \vec{r_2})} \quad -①$$

rel. prob coeff of a plane wave state where one e^- is in $\hbar\vec{k}$ + the other has momentum $-\hbar\vec{k}$.

Effect of the other N-2 particles

Since all $k < k_F$ states are occupied,

$$g(\vec{k}) = 0 \quad \text{for } |\vec{k}| < k_F$$



Schrodinger eqn.

$$-\frac{\hbar^2}{2m}\left[\Delta_1 + \Delta_2\right]\psi + V(r_1, r_2)\psi = (E + 2E_F)\psi \quad -②$$

$$2E_F = \frac{\hbar^2 k_F^2}{m}.$$

E is measured from the Fermi energy.

$$x^n. \quad V(\bar{r_1}, \bar{r_2}) = V(\bar{r_1} - \bar{r_2}) = \sum_k V_k e^{i\vec{k}(\bar{r_1} - \bar{r_2})} \quad -③$$

Using ① + ③ in ②, we obtain for each plane wave component

$$\frac{\hbar^2 k^2}{m} g(\vec{k}) + \sum_{\vec{k}'} V_{\overline{k-k'}} g(\vec{k}') = (2E_F + E) g(\vec{k})$$

---

* spin singlet state + centre of mass at rest.

$$g(k) = g(-k) \qquad \sum_k |g_k|^2 = 1$$

To get an analytical soln. we assume a toy interaction

$$V_{\bar{k},\bar{k}'} = -V \quad \text{if } E_f < \frac{\hbar^2 k^2}{2m} < E_f + \hbar\omega_D$$

$$\& \quad E_f < \frac{\hbar^2 k'^2}{2m} < E_f + \hbar\omega_D$$

$$= 0 \quad \text{otherwise.}$$

$\omega_D \rightarrow$ Debye freq.

$$-V \sum_{|k'| > k_f} g(\bar{k}') = C.$$

$$\Rightarrow \quad g(\bar{k}) = \frac{C}{E + 2E_f - \frac{\hbar^2 k^2}{2m}}.$$

Self consistency demands that

$$-VC \sum_{|k'| > k_f} \frac{1}{E + 2E_f - \frac{\hbar^2 k'^2}{2m}} = C.$$

$$1 = V \sum_{k'} \frac{1}{\frac{\hbar^2 k'^2}{2m} - E - 2E_f}$$

Sum is ~~effected~~ over the annulus of width ω_D. $d\varepsilon = 2k\,dk$, $\varepsilon = k^2$

Using $\xi = \frac{\hbar^2 k^2}{2m} - E_f \rightarrow$ single particle energy $\int k^{d-1}\,dk$ & ε

density of states $N(\xi) = 2\left.\frac{4\pi k^2}{(2\pi)^3}\frac{dk}{d\xi}\right]_{E_F}$ $d\varepsilon = 2k\,dk$, $\frac{1}{k}$

we have $\frac{1}{V} = \int_0^{\hbar\omega_D} \frac{1}{2\xi - E} N(\xi)\,d\xi.$ $\rightarrow \frac{\varepsilon}{\sqrt{\varepsilon}}$

Assuming $\omega_D \ll E_f$, $N(\xi) = \text{cste} \equiv$ fermi energy value $N(0)$ –

$$1 = \frac{N(0)V}{2} \ln \frac{E - 2\hbar\omega_D}{E}$$

$$\left.\frac{1}{2} \ln \xi - \frac{E}{2}\right|_0^{\hbar\omega_D}$$

$$\frac{1}{2} \ln \frac{\hbar\omega_D + E/2}{E/2}$$

If the interaction is very weak
$N(0)V \ll 1$ then,

$$E = -2\hbar\omega_D \exp - \frac{2}{N(0)V}$$

$\Longrightarrow$ weak bound state of 2 e^{-}s. for arbitrarily weak attract. X^n!

(not always true in 3D ... ^{2}He molecule is unbound inspite of $-\frac{1}{r^6}$ vd Waals X^n.)

Pair state is in a zero momentum state.

If we take into account the spin, the solns. to the eqns. are angular momentum eigenstates, so $g(k)$ should have definite parity. $g(k) = g(-k)$ or $g(k) = -g(-k)$.

We consider $g(k) = g(-k)$ & this corresponds to symm. spatial fn. & hence a spin singlet state to restore antisymm nature of fermion wavefn.

$\longrightarrow$ Pair state having finite $q \rightarrow$ bound state only for exponentially small q.

$$E \rightarrow E + v_F|q|/2 \nearrow$$

What does the wave fn. look like?
(s-wave orbital state)

$$\psi(r) = \frac{1}{r}\frac{\partial}{\partial r}\int_{k_F}^{k_D} \frac{\cos kr}{2\varepsilon_k + |E|} \qquad \varepsilon_k = \frac{\hbar^2(k^2 - k_F^2)}{2m} = \frac{\hbar^2 k^2}{2m} - E_F$$

$k_D \rightarrow$ cut off wave vector &
$r = |\vec{r}_1 - \vec{r}_2|$.

General structure of $\psi \longrightarrow \Sigma$ terms of $(\cos k_F r, \sin k_F r) \times$ decreasing fns. of r

$\frac{1}{r}$ for small r. $\frac{1}{r^2}$ for larger r.

Cross over happens at $r \sim \frac{\hbar v_F}{|E|} \sim \frac{\hbar v_F}{\hbar\omega_D} \exp\frac{2}{N(0)V} \equiv \xi_c$.

$v_F = \frac{\hbar k_F}{m}$

The bound state has a radius ξ_c in the sense prob of finding particles at $r >> \xi_c \rightarrow 0$ as $\frac{1}{r}$

But for small r $\psi(r) \sim \frac{\sin k_F r}{k_F r}$.

Generalize to finite T!
assume that pair can only occupy states $k\uparrow, -k\downarrow$ if $N-2$ e^-s do not occupy them.
At finite T, probability for occupation of this pair is

$$\frac{1}{\left(1+e^{+\beta\varepsilon_k}\right)^2}.$$

Replacing $\rho(\varepsilon) = N(0)\,\theta(\varepsilon) \longrightarrow \frac{N(0)}{(1+e^{\beta\varepsilon})^2}$

$$\frac{1}{V} = N(0)\int_{-\infty}^{\infty} \frac{d\varepsilon}{(2\varepsilon - E)(1+e^{+\beta\varepsilon})^2} \qquad \text{— (A)}$$

But ε can have any sign at finite T.

Singularity at $\varepsilon = 0$ is removed!
equivalent to replacing lower limit by $k_B T$ (not 0!)
(A) has no $E<0$ soln. if condition not satisfied

$$\int_{k_B T}^{\hbar\omega_D} \frac{d\varepsilon}{2\varepsilon} > \frac{1}{N(0)V}$$

$$\ln \frac{\hbar\omega_D}{k_B T} = \frac{2}{N(0)V}$$

$$\frac{k_B T}{\hbar\omega_D} = e^{-\frac{2}{N(0)V}}$$

i.e. above a critical T_c

T_c ~~$= \hbar\omega_D$~~

$$T_c \sim \frac{\hbar\omega_D}{k_B} \exp - \frac{2}{N(0)V}$$ [order of mag estimates

3 features

- exponentially small energy of bound state
- " large radius of pair
- critical temp.

reflected in many body BCS state!

Origin of attractive interactions

Repulsion energy between $V(q) = \dfrac{e^2}{4\pi \varepsilon_0 |\vec{q}|^2}$.

(in vacuum)

$\vec{k}$ $\quad$ $-\vec{k}$ $\longrightarrow$

$\vec{k}'$ $\quad$ $-\vec{k}'$ $\qquad \vec{k}' = \vec{k} + \vec{q}$

But interactions also screen Coulomb .xn.

+ interaction with +ve ions of system.

$\Rightarrow$ Polarization $\vec{P}$ of environment.

$\Rightarrow$ electrical induction $\vec{D} = \varepsilon_0 (\vec{E} + \vec{P})$ $\hookrightarrow$ electrical displacement

Since $\vec{P}$ & $\hat{E}$ are collinear,

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$$

$\vec{P} = (\varepsilon_r - 1)\vec{E}$ $\rightarrow$ $\varepsilon_r \rightarrow$ relative dielectric constant of medium

$\Rightarrow$ Screening! $\quad V_q \sim \dfrac{1}{q^2} \qquad V_q = \dfrac{1}{q^2 + \lambda_s^{-2}} \quad \dfrac{e^{-\lambda_s^{-1} r}}{r}$

$$V_{\vec{q}} = \frac{e^2}{4\pi \varepsilon_r(\vec{q},\omega)\, \varepsilon_0 |\vec{q}|^2}.$$

$\varepsilon_r(\vec{q},\omega) \longrightarrow$ dielectric response of system when we insert an ext. charge whose density has wave vector $\vec{q}$ or varies in time with freq. ω. [$\vec{E}$ field varies with the wave length that field

$$\vec{E}(q,\omega) = \vec{E}_q e^{i(\vec{q}\cdot\vec{r}+\omega t)}$$

~~All +ve~~ Every +ve ion is now displaced —

$$\vec{\xi}(\vec{q},\omega) = \vec{\xi}_q e^{i(\vec{q}\cdot\vec{r}+\omega t)}$$

Classical mechanics dictates

$$M\frac{d^2\vec{\xi}}{dt^2} = -\alpha_{\vec{q}}\,\vec{\xi} + e\vec{E} \rightarrow \text{electric drift.}$$

elastic restoring force.

(monovalent with charge +e)

M — mass of ion.

In Fourier space)

$$-M\omega^2\vec{\xi}_q = -\alpha_{\vec{q}}\vec{\xi}_{\vec{q}} + e\vec{E}_{\vec{q}}$$

Polarisation correspond to $\vec{\xi}_{\vec{q}}$ is

$$\vec{P}_q^{\,ion} = \frac{n_{ion}}{\varepsilon_0} e\,\vec{\xi}_{\vec{q}}$$

$n_{ion} \rightarrow$ density of ions.

$$\vec{P}_{\vec{q}}^{\,ion} = \frac{n_{ion}}{\varepsilon_0}\cdot\frac{e^2\vec{E}_{\vec{q}}}{\alpha_q - M\omega^2}$$

$\omega_p \rightarrow$ plasma freq of ions

$$= \frac{\omega_p^2}{\Omega^2-\omega^2}\vec{E}_{\vec{q}} \qquad \omega_p^2 = \frac{n_{ion}e^2}{M\varepsilon_0}.$$

$$\Omega = \sqrt{\frac{\alpha_{\vec{q}}}{M}} \rightarrow \text{phononic frequency (elastic forces)}$$

Electronic polarization

Thomas Fermi

$$\vec{P}^{el}_{\vec{q}} = \frac{\omega_p^2 \, \vec{E_q}}{\omega_p^2 \lambda_s^2 q^2 - \omega^2}$$

$\lambda_s \to$ screening length for e^-

$\lambda_s \sim a$

$\lambda_s q \sim \frac{1}{[illegible]}$

neglecting ω^2

$$\vec{P}^{el}_{\vec{q}} = \frac{\vec{E_q}}{\lambda_s^2 q^2}$$

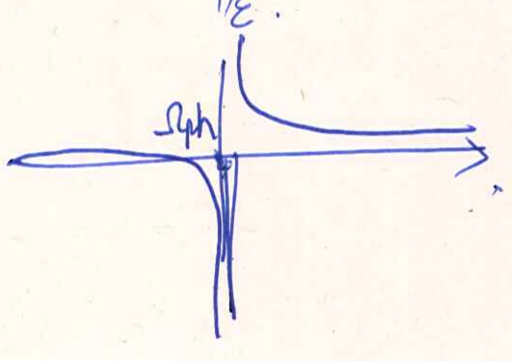
(Thomas-fermi approx)

$$\vec{P} = \vec{P}^{ion} + \vec{P}^{el}$$

$$\varepsilon_r(q,\omega) - 1 = \frac{1}{q^2 \lambda_s^2} + \frac{\omega_p^2}{\Omega^2 - \omega^2}$$

$$\frac{1}{\varepsilon_r(\vec{q},\omega)} = \frac{\Omega^2 - \omega^2}{\omega_p^2 + \varepsilon_{el}(\Omega^2 - \omega^2)}$$

$$\varepsilon_{el} = 1 + \frac{1}{q^2 \lambda_s^2}$$



$$\Omega_{ph} = \left[\Omega^2 + \omega_p^2/\varepsilon_{el}\right]^{1/2}$$

$\Omega < \omega < \Omega_{ph}$

ε_r is < 0

Attraction is stronger if $\Omega \to 0$.

as a result $V(q,\omega)$ can be negative.