Exercise 1. *Entanglement and teleportation*

Imagine that Alice (A) has a pure state $|\psi\rangle_S$ *of a system* S *in her lab. She wants to send that state to Bob, who lives, of course, on the Moon, but she does not trust the postwoman Eve to carry it there personally. We have seen that if Alice and Bob share an entangled state Alice can "teleport" the state* $|\psi\rangle$ *to the system* B *that Bob controls by.*

Formally, we have three systems $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$*. In this exercise we will assume all three are qubits. The initial state is*

$$|\psi\rangle_S \otimes \frac{1}{\sqrt{2}} \left(|00\rangle_{AB} + |11\rangle_{AB} \right), \tag{1}$$

i.e. S *is decoupled from* A *and* B *and these two are fully entangled in a Bell state. We may write* $|\psi\rangle_S = \alpha|0\rangle_S + \beta|1\rangle_S$.

(a) *In a first step, Alice will measure systems* S *and* A *jointly in the Bell basis,*

$$\tfrac{1}{\sqrt{2}} \left(|00\rangle_{SA} + |11\rangle_{SA} \right), \tfrac{1}{\sqrt{2}} \left(|00\rangle_{SA} - |11\rangle_{SA} \right), \tfrac{1}{\sqrt{2}} \left(|01\rangle_{SA} + |10\rangle_{SA} \right), \tfrac{1}{\sqrt{2}} \left(|01\rangle_{SA} - |10\rangle_{SA} \right).$$

Then Alice communicates (classicaly) the result of her measurement to Bob. What is the reduced state of Bob's system (B) for each of the possible outcomes?

(b) *Depending on the outcome of the measurement by Alice, Bob may have to perform certain unitary operations on his qubit so that he recovers* $|\psi\rangle$*. Which operations are these?*

(c) *Suppose that Alice does not manage to tell Bob the outcome of her measurement. Show that in this case he does not have any information about his reduced state and therefore does not know which operation to apply in order to obtain* $|\psi\rangle$.

(d) *Show that this method of quantum teleportation also works for mixed states* ρ_S.

(e) *There is no reason why the state* ρ_S *cannot be entangled with some other system that Alice and Bob do not control. Consider a purification of* ρ_S *on a reference system* R*, i.e.* $|\phi\rangle_{SR}$ *s.t.*

$$\rho_S = \mathrm{tr}_R |\phi\rangle\langle\phi|_{SR}. \tag{2}$$

Show that if you apply the quantum teleportation protocol on $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$*, not touching the reference system, the final state on* $\mathcal{H}_B \otimes \mathcal{H}_R$ *is* $|\phi\rangle_{BR}$.

This implies that quantum teleportation preserves correlations (including entanglement) – it simply transfers it from S *and* R *to* B *and* R.

Solution.

(a) First let us compute the probability that Alice obtains outcome k when she measures the joint state of S and A. The density operator $|\psi\rangle\langle\psi|$ can be represented in the computational basis as

$$|\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}. \tag{S.1}$$

Denote the original state shared by Alice and Bob by $|ab^1\rangle_{AB}$. The reduced state on Alice's qubit is fully mixed, $\rho_A = \mathrm{tr}_B[|ab^1\rangle\langle ab^1|] = \mathbb{1}_A/2$. The joint state of S and A is then

$$\rho^0_{SA} = \mathrm{tr}_B(|\psi\rangle\langle\psi|_S \otimes |ab^1\rangle\langle ab^1|_{AB}) = |\psi\rangle\langle\psi|_S \otimes \frac{\mathbb{1}_A}{2} = \begin{pmatrix} |\alpha|^2 & 0 & \alpha\beta^* & 0 \\ 0 & |\alpha|^2 & 0 & \alpha\beta^* \\ \alpha^*\beta & 0 & |\beta|^2 & 0 \\ 0 & \alpha^*\beta & 0 & |\beta|^2 \end{pmatrix} \tag{S.2}$$

in the computational basis. For commodity, we can rewrite this state in the Bell basis,

$$\rho^0_{SA} = U^\dagger \, \frac{1}{4} \begin{pmatrix} 1 & 2|\alpha|^2-1 & \alpha^*\beta+\alpha\beta^* & \alpha^*\beta-\alpha\beta^* \\ 2|\alpha|^2-1 & 1 & -\alpha^*\beta+\alpha\beta^* & -\alpha^*\beta-\alpha\beta^* \\ \alpha^*\beta+\alpha\beta^* & \alpha^*\beta-\alpha\beta^* & 1 & 2|\alpha|^2-1 \\ -\alpha^*\beta+\alpha\beta^* & -\alpha^*\beta-\alpha\beta^* & 2|\alpha|^2-1 & 1 \end{pmatrix} U\,, \tag{S.3}$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}. \tag{S.4}$$

Now it is easy to see that the probability of obtaining each of the outcomes is the same. Only the diagonal terms of ρ^0_{SA} in the Bell basis matter, and these are all $1/4$. For instance, the probability of getting 1 is

$$\begin{aligned} \mathrm{Pr}_1 &= \mathrm{tr}\left[|sa^1\rangle\langle sa^1|\; \rho^0_{SA}\right] \\ &= \frac{1}{4}\,\mathrm{tr}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2|\alpha|^2-1 & \alpha^*\beta+\alpha\beta^* & \alpha^*\beta-\alpha\beta^* \\ 2|\alpha|^2-1 & 1 & -\alpha^*\beta+\alpha\beta^* & -\alpha^*\beta-\alpha\beta^* \\ \alpha^*\beta+\alpha\beta^* & \alpha^*\beta-\alpha\beta^* & 1 & 2|\alpha|^2-1 \\ -\alpha^*\beta+\alpha\beta^* & -\alpha^*\beta-\alpha\beta^* & 2|\alpha|^2-1 & 1 \end{pmatrix}\right] \\ &= \frac{1}{4}. \end{aligned} \tag{S.5}$$

The global state of the system after the measurement on S and A will be pure, because the initial global state was pure too. It is given by

$$|\Omega^k_\psi\rangle_{SAB} = \frac{1}{\sqrt{\mathrm{Pr}_k}}\Big(|sa^k\rangle\langle sa^k| \otimes \mathbb{1}_B\Big)|\psi\rangle \otimes |ab^1\rangle$$

(we could have also used the expression for the final state in terms of density operators, and not pure states). See (b) for a list of the states after each measurement outcome, and the tips for a step-by-step derivation.

(b) Here is a table with the reduced states on Alice's and Bob's sides and the operations that Bob has to perform on his state in order to recover $|\psi\rangle$, according to the outcomes of

Alice's measurement.

Alice's outcome	Alice's state	Bob's state	Bob performs
1	$\lvert sa^1\rangle = \frac{1}{\sqrt{2}}\left(\lvert 0_S 0_A\rangle + \lvert 1_S 1_A\rangle\right)$	$\lvert b_\psi^1\rangle = \alpha\lvert 0\rangle + \beta\lvert 1\rangle$	$O_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
2	$\lvert sa^2\rangle = \frac{1}{\sqrt{2}}\left(\lvert 0_S 0_A\rangle - \lvert 1_S 1_A\rangle\right)$	$\lvert b_\psi^2\rangle = \alpha\lvert 0\rangle - \beta\lvert 1\rangle$	$O_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
3	$\lvert sa^3\rangle = \frac{1}{\sqrt{2}}\left(\lvert 0_S 1_A\rangle + \lvert 1_S 0_A\rangle\right)$	$\lvert b_\psi^3\rangle = \beta\lvert 0\rangle + \alpha\lvert 1\rangle$	$O_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
4	$\lvert sa^4\rangle = \frac{1}{\sqrt{2}}\left(\lvert 0_S 1_A\rangle - \lvert 1_S 0_A\rangle\right)$	$\lvert b_\psi^4\rangle = \beta\lvert 0\rangle - \alpha\lvert 1\rangle$	$O_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(c) Alice has a probability $\Pr_k = 1/4$ of obtaining each of the different outcomes. If Bob does not know which one Alice got, his knowledge of the state of his system can be represented by a classical state,

$$\rho_B = \sum_k \Pr_k |b_\psi^k\rangle\langle b_\psi^k| \tag{S.6}$$

$$= \frac{1}{4}\sum_k |b_\psi^k\rangle\langle b_\psi^k| \tag{S.7}$$

$$= \frac{1}{4}\left[\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} + \begin{pmatrix} |\alpha|^2 & -\alpha\beta^* \\ -\alpha^*\beta & |\beta|^2 \end{pmatrix} + \begin{pmatrix} |\beta|^2 & \alpha^*\beta \\ \alpha\beta^* & |\alpha|^2 \end{pmatrix} + \begin{pmatrix} |\beta|^2 & -\alpha^*\beta \\ -\alpha\beta^* & |\alpha|^2 \end{pmatrix}\right] \tag{S.8}$$

$$= \frac{1}{2}\begin{pmatrix} |\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\alpha|^2 + |\beta|^2 \end{pmatrix} = \frac{\mathbb{1}}{2}, \tag{S.9}$$

because $|\alpha|^2 + |\beta|^2 = 1$, since $|\psi\rangle$ is normalised. The state of Bob's qubit is fully mixed, which means that he has no information about its state. Alice, however, knows perfectly the state of Bob's qubit, and represents it as a pure state, because she knows the outcome of her measurement.

(d) First we compute the probability of Alice obatining outcome k if she measures the joint state of S and A in the Bell basis. If

$$\rho_S = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \tag{S.10}$$

in the computational basis, then, in the Bell basis,

$$\rho_{SA} = \frac{1}{4}\begin{pmatrix} 1 & -1+2a & b+c & -b+c \\ -1+2a & 1 & b-c & -b-c \\ b+c & -b+c & 1 & -1+2a \\ b-c & -b-c & -1+2a & 1 \end{pmatrix}, \tag{S.11}$$

so again the probability of obtaing each of the different outcomes is $1/4$.

Now we will see what the final state of the global system is. We can expand ρ_S in its eigenbasis, $\rho_S = \sum_i c_i |s_i\rangle\langle s_i|$ (in this case, because S is a qubit, $i = 1, 2$). The state of the global system after the measurement is

$$\begin{aligned}
\rho^k_{SAB} &= \frac{1}{\Pr_k}\left[(|sa^k\rangle\langle sa^k|\otimes \mathbb{1}_B)\rho^0_{SAB}(|sa^k\rangle\langle sa^k|\otimes \mathbb{1}_B)\right] \\
&= \frac{1}{4}\left[(|sa^k\rangle\langle sa^k|\otimes \mathbb{1}_B)\left(\left[\sum_i c_i|s_i\rangle\langle s_i|\right]\otimes|ab^1\rangle\langle ab^1|\right)(|sa^k\rangle\langle sa^k|\otimes \mathbb{1}_B)\right] \\
&= \sum_i c_i\left(\frac{1}{4}\left[(|sa^k\rangle\langle sa^k|\otimes \mathbb{1}_B)\left(|s_i\rangle\langle s_i|\otimes|ab^1\rangle\langle ab^1|\right)(|sa^k\rangle\langle sa^k|\otimes \mathbb{1}_B)\right]\right).
\end{aligned} \tag{S.12}$$

What is written in blue is the pure state in which the system would end up if the initial state of system S was $|s_i\rangle$, $|\Omega^k_{s_i}\rangle_{SAB}$. The reduced state on Bob's qubit is

$$\begin{aligned}
\rho^k_B &= \mathrm{tr}_{SA}(\rho_{SAB^k}) \\
&= \mathrm{tr}_{SA}\left(\sum_i c_i|\Omega^k_{s_i}\rangle\langle\Omega^k_{s_i}|_{SAB}\right) \\
&= \sum_i c_i\ \mathrm{tr}_{SA}\left(|\Omega^k_{s_i}\rangle\langle\Omega^k_{s_i}|_{SAB}\right) \\
&= \sum_i c_i\ |b^k_{s_i}\rangle\langle b^k_{s_i}|.
\end{aligned} \tag{S.13}$$

If Bob applies the unitary operation O_k on his qubit, he will recover ρ_S,

$$\begin{aligned}
O_k\rho^k_B {O_k}^\dagger &= O_k\left(\sum_i c_i|b^k\rangle\langle b^k|_{s_i}\right){O_k}^\dagger \\
&= \sum_i c_i\ \left(O_k|b^k_{s_i}\rangle\langle b^k_{s_i}|{O_k}^\dagger\right) \\
&= \sum_i c_i|s_i\rangle\langle s_i| = \rho_S.
\end{aligned} \tag{S.14}$$

On the other hand, the reduced state of S and A after the protocol is $|sa^k\rangle$, as before.

(e) A Schimdt decomposition of the pure joint state of S and R gives us $|\phi\rangle_{RS} = \sum_i c_i|r_i\rangle_R\otimes|s_i\rangle_S$. The inital global state of R, S, A and B is

$$\begin{aligned}
|\Omega^0_\phi\rangle_{RSAB} &= |\phi\rangle_{RS}\otimes|ab^1\rangle_{AB} \\
&= \left(\sum_i c_i|r_i\rangle_R\otimes|s_i\rangle_S\right)\otimes|ab^1\rangle_{AB} \\
&= \sum_i c_i\left(|r_i\rangle_R\otimes|s_i\rangle_R\otimes|ab^1\rangle_{AB}\right).
\end{aligned} \tag{S.15}$$

The protocol does not affect R. The final global state is

$$|\Omega^k_\phi\rangle_{RSAB} = \sum_i c_i\left(|r_i\rangle_R\otimes|sa^k\rangle_{SA}\otimes|s_i\rangle_B\right) \tag{S.16}$$

The reduced state of R and B is

$$\sum_i c_i|r_i\rangle_R\otimes|s_i\rangle_B = |\phi\rangle_{RB}. \tag{S.17}$$

Exercise 2. ***Probabilistic entanglement generation using post-selection***

Suppose we have two atoms that we would like to entangle. How can we do this? In this exercise we describe one way of doing so using post-selection. Call the identical atoms A and B. We are only interested in two energy levels of each atom, which we denote by $|g\rangle$ (ground state) and $|e\rangle$ (excited state). From now on we treat them as qubits.

(a) *Assume that both atoms are initially in the excited state, $|\Psi_{\text{in}}\rangle_{AB} = |e\rangle_A|e\rangle_B$. Due to spontaneous emission the atoms may decay to the ground state. Taking the electromagnetic field (emf) into account, what is the state of the total system ($A \otimes B \otimes \text{emf}$) after some time?*

Hint: *Assume that the probability for each atom to decay during this time is ε and that this happens independently for both atoms. Also, split the emf into the relevant modes.*

(b) *If you can make arbitrary measurements on the emf, i.e. on the emitted photons if there are any, find a measurement that, if the right outcome is obtained, leaves the atoms in a maximally entangled state. What is the probability that this outcome occurs?*

Remark: *This is called post-selection. Depending on the outcome of the measurement we know that the atoms are entangled (and we can do interesting things with them) or not. In the latter case the protocol would be aborted.*

(c) *Can you come up with an experimental setup to perform the proposed measurement?*

Solution.

(a) The total system, in sloppy notation written as $A \otimes B \otimes \text{emf}$, is closed, hence the initially pure state $|e\rangle_A|e\rangle_B|0\rangle_{\text{emf}}$ will evolve unitarily to another pure state. Here, $|0\rangle_{\text{emf}}$ denotes the vacuum state of the electromagnetic field. The field consists of many modes which are not occupied initially, hence we can be more precise by writing $|0\rangle_{\text{emf}} = |0\rangle_a|0\rangle_b|0\rangle_c\cdots$, where $a, b, c, \ldots$ are the different modes.

Spontaneous emission now changes the total state to

$$|\Psi_{\text{spont}}\rangle_{ABabc\cdots} = \big(\sqrt{1-\varepsilon}\,|e\rangle_A|0\rangle_a + e^{i\phi}\sqrt{\varepsilon}\,|g\rangle_A|1\rangle_a\big) \otimes \big(\sqrt{1-\varepsilon}\,|e\rangle_B|0\rangle_b + e^{i\theta}\sqrt{\varepsilon}\,|g\rangle_B|1\rangle_b\big) \otimes |0\rangle_c\cdots \tag{S.18}$$

where ϕ and θ are some phases[1] and we assume that atom A can emit photons in mode a, while atom B emits photons in mode b.

(b) A measurement of the photons in modes a and b in the Bell basis is a good choice. By this we mean the basis

$$\{\,|00\rangle + |11\rangle\,, |00\rangle - |11\rangle\,, |10\rangle + |01\rangle\,, |10\rangle - |01\rangle\,\}\,. \tag{S.19}$$

So one can think of the Bell states here as states of photons that are entangled 'with the vacuum'. In particular, when the outcome corresponding to $\frac{1}{\sqrt{2}}(|10\rangle_{ab}+|01\rangle_{ab})$ is obtained, we know that the (yet unnormalized) post-measurement state of the atoms is

$$\frac{1}{\sqrt{2}}(\langle 10|_{ab} + \langle 01|_{ab})|\Psi_{\text{spont}}\rangle_{ABab} = \frac{\sqrt{1-\varepsilon}\sqrt{\varepsilon}}{\sqrt{2}}\big(e^{i\phi}|ge\rangle_{AB} + e^{i\theta}|eg\rangle_{AB}\big)\,, \tag{S.20}$$

[1] The phases are constant when repeating the process many times and usually known to the experimenter. If not, the final state would not be maximally entangled but mixed!

a maximally entangled state. Likewise, if the outcome corresponding to $\frac{1}{\sqrt{2}}(|10\rangle_{ab} - |01\rangle_{ab})$ is obtained the final post-measurement state of the atoms is

$$\frac{\sqrt{1-\varepsilon}\sqrt{\varepsilon}}{\sqrt{2}}\left(e^{i\phi}|ge\rangle_{AB} - e^{i\theta}|eg\rangle_{AB}\right). \tag{S.21}$$

From the normalization of the states we can read out the probability to obtain one of them. In this case it equals $(1-\varepsilon)\varepsilon \overset{\varepsilon \ll 1}{\approx} \varepsilon$ for each of the desired outcomes, so 2ε in total. Hence, due to the small probability to observe spontaneous emission, the probability to post-select a maximally entangled state is rather small. Nevertheless this process is used to entangle qubits, for instance in *spontaneous parametric down conversion.*

(c) What comes here is an idealised scenario. However, we believe it still helps to understand how such a measurement could be implemented.

Suppose we place a 50/50 beam splitter next to the two atoms as depicted below. We denote the ingoing modes by a, b and the outgoing modes by c, d. After the beam splitter detectors (depicted as half circles C and D) are placed which tick if they measure a photon. In the language of second quantization the unitary evolution of an incoming 1-photon state due to the beam splitter is given by

$$U_{BS} = \frac{1}{\sqrt{2}}\left((c^\dagger + d^\dagger)a + (c^\dagger - d^\dagger)b\right), \tag{S.22}$$

where $a, a^\dagger, b, b^\dagger, \ldots$ are the annihilation and creation operators of the corresponding modes. Hence we find the following state transformations:

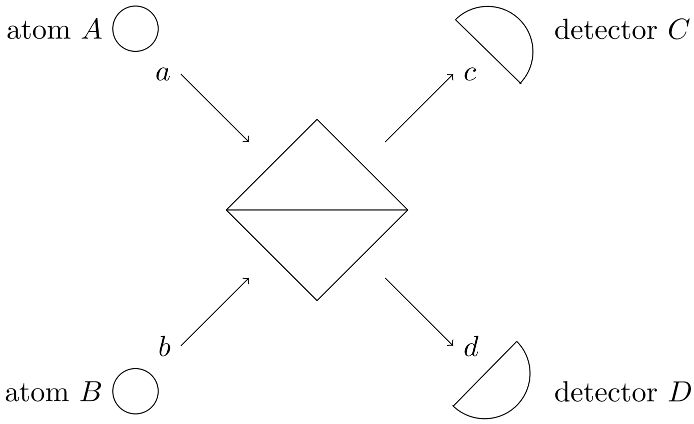
$$\begin{aligned} \frac{1}{\sqrt{2}}\left(|10\rangle_{ab} + |01\rangle_{ab}\right) &\overset{U_{BS}}{\longmapsto} |10\rangle_{cd}, \\ \frac{1}{\sqrt{2}}\left(|10\rangle_{ab} - |01\rangle_{ab}\right) &\overset{U_{BS}}{\longmapsto} |01\rangle_{cd}. \end{aligned} \tag{S.23}$$

Thus, detecting exactly one photon in mode c and none in mode d after the beam splitter amounts to measuring the initial photons in mode a and b in the state $\frac{1}{\sqrt{2}}(|10\rangle_{ab} + |01\rangle_{ab})$ and likewise for the other maximally entangled 1-photon state.

Notice that the above analysis was conducted for a 1-photon state and does not hold for other configurations.[2] In the case of the 0-photon state (the vacuum) no detector will click. So this will not be a problem. However, if both atoms emit a photon and the incoming state has support on $|11\rangle_{ab}$ it can happen that one of the photons is reflected by the beam splitter, while the other one passes. The state after the beam splitter can then have support on $|02\rangle_{cd}$ or $|20\rangle_{cd}$. In these cases, two photons hit one detector. Photon detectors are usually not able to distinguish the number of photons that hit them simultaneously and therefore, if two photons arrive at the same detector, we only have one click and might mistakenly take this as one of the outcomes corresponding to the maximally entangled 1-photon states.

By choosing the time between preparing the atoms and measuring the photons appropriately one can make ε arbitrarily small. The probability to measure a one photon state then decreases as ε, while the probability to measure a 2-photon state is suppressed by ε^2. So there is a trade-off between the probability of measuring a photon state that leaves the atoms in an entangled state and the probability of mistakenly taking a single detector click as one of the desired outcomes. But by accepting a very low probability of entangling the atoms one can make the probability of a mistake arbitrarily small.

[2] You can see that for states outside this subspace, e.g. $|00\rangle_{ab}$, the 'unitary' in (S.22) is no unitary anymore.



Exercise 3. ***Gibbs paradox***

(a) *Consider a container consisting of two compartments separated by a removable wall (left figure below). The two compartments are filled with an ideal gas at the same pressure and temperature. Assume first that you are not aware of any process that distinguishes the gas in the left compartment from that in the right one. By how much does the entropy of the container change when you remove the wall?*

(b) *Assume now that you discover two materials, A and B, and that you find that the gas in the left compartment can pass through A but not through B, and that the gas in the right compartment can pass through B but not through A. The existence of two materials with those characteristics shows, in particular, that the two compartments are actually filled with two different types of gas (right figure below). By how much does the entropy of the container now change when removing the wall? Carry out this calculation both in the framework of Phenomenological Thermodynamics and in Statistical Mechanics.*

Hint: *In order to calculate the entropy change in the framework of Phenomenological Thermodynamics you should try to find a reversible process resulting in the same state change and use* $\mathrm{d}S = \delta Q_{\text{rev}}/T$, *where* Q_{rev} *is the reversibly exchanged heat and* T *is the temperature of the heat bath.*

(c) *Explain the different outcomes of the entropy changes in the above scenarios.*

Solution.

(a) Under the given assumptions, removing the wall is a reversible process (reinserting it brings the system back to its initial state – if it did not you could use this process to distinguish the types of gas in the two compartments). In addition, it is adiabatic, i.e. no heat is supplied to the system. Therefore, the entropy change equals 0.

(b) Removing the wall is now no longer a reversible process (after reinserting it the system is in a different state). In order to calculate the entropy change within the framework of Phenomenological Thermodynamics, we need to find a reversible process that results in the same transformation of state as removing the wall. An example of such a process is the following: we use materials A and B as semi-permeable membranes to mix the gases reversibly and isothermally, thereby changing the initial total volume of each gas, say $V_0/2$, to V_0. This process has work cost of

$$W = 2\int_{V_0/2}^{V_0} -pdV = 2\int_{V_0/2}^{V_0} -\frac{\frac{1}{2}RT}{V}dV = -RT(\ln V_0 - \ln V_0/2) = -RT\ln 2\ , \tag{S.24}$$

where R is the gas constant and where we assumed that we have 1 mol of gas in total. (Note that W is negative, corresponding to an actual work gain.) The work has to be compensated by a heat flow $Q = -W$ into the system. According to the definition of entropy, this is equivalent to an entropy change of $S = Q/T = R\ln 2$.

Within Statistical Mechanics, the same conclusion can be reached via the following simple observation. For any configuration consisting of N particles in total there are $\binom{N}{N/2}$ configurations with $N/2$ particles of one type and $N/2$ particles of the other type. Hence, in the latter case, the entropy is larger by an amount of

$$S = k_B \ln\binom{N}{N/2} \approx k_B\ln\left(2^N/\sqrt{\pi N/2}\right) = Nk_B\ln 2 - k_B\ln\sqrt{\pi N/2} \approx Nk_B\ln 2\ , \tag{S.25}$$

where we have used Stirling's approximation. If we let N be the Avogadro number then $Nk_B = R$, corresponding to the result obtained within Phenomenological Thermodynamics for 1 mol of gas.

(c) Two agents of which only one knows about materials A and B will in fact assign different entropy changes to the processes. However, this is not a problem. How entropy is defined depends on our knowledge about the system at hand – the definition is subjective in this sense.

Thinking of Phenomenological Thermodynamics, one usually starts by writing down all relevant macroscopic quantities that can be manipulated. Depending on this, it is possible to decide which processes are *working processes* and which are not. This will then be used to define work δA, which is in turn used to define heat $\delta Q = \mathrm{d}U - \delta A$. Thermodynamic entropy is defined by $\mathrm{d}S = \delta Q_{\mathrm{rev}}/T$, hence the 'choice' of macroscopic quantities that can be manipulated has an influence on what entropy is. In the example at hand, the agent in possession of materials A and B will say that separating the mixed gases is a working process, whereas the other will not.

Likewise entropy is subjective in the framework of Statistical Mechanics. Ignoring the distinguishability of the two kinds of particles will give us an entropy increase of $S = 0$. Also here, the knowledge about the system at hand has an influence on the definition of entropy.