Exercise 1. *Depolarizing channel*

We are given two two-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and a completely positive trace preserving (CPTP) map $\mathcal{E}_p : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$, $0 \leq p \leq 1$, defined as

$$\mathcal{E}_p(\rho) = p\frac{\mathbb{1}}{2} + (1-p)\rho. \tag{1}$$

(a) *An operator-sum representation (also called the Kraus-operator representation) of a CPTP map $\mathcal{E} : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ is a decomposition $\{E_k\}_k$ of operators $E_k \in Hom(\mathcal{H}_A, \mathcal{H}_B)$, $\sum_k E_k {E_k}^\dagger = \mathbb{1}$, such that*

$$\mathcal{E}(\rho) = \sum_k E_k \rho {E_k}^\dagger.$$

Find an operator-sum representation for $\mathcal{E}_p$.

Hint: *Remember that $\rho \in \mathcal{S}(\mathcal{H}_A)$ can be written in the Bloch sphere representation:*

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}), \quad \vec{r} \in \mathbb{R}^3, \quad |\vec{r}| \leq 1, \quad \vec{r}\cdot\vec{\sigma} = r_x\sigma_x + r_y\sigma_y + r_z\sigma_z, \tag{2}$$

where σ_x, σ_y and σ_z are Pauli matrices. It may be useful to show that

$$\mathbb{1} = \frac{1}{2}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z).$$

(b) *What happens to the radius $\vec{r}$ when we apply $\mathcal{E}_p$? How can this be interpreted?*

(c) *A probability distribution $P_A(0) = q$, $P_A(1) = 1-q$ can be encoded in a quantum state on $\mathcal{H}_A$ as $\hat{\rho} = q|0\rangle\langle 0|_A + (1-q)|1\rangle\langle 1|_A$. Calculate $\mathcal{E}(\hat{\rho})$ and the conditional probabilities $P_{B|A}$ as well as P_B after measuring $\mathcal{E}(\hat{\rho})$ in the standard basis $\{|0\rangle_B, |1\rangle_B\}$.*

Solution.

(a) For simplicity of notation, we denote the Pauli matrices by $\sigma_x = X, \sigma_y = Y, \sigma_z = Z$.

Defining $A := \frac{1}{2}(\rho + X\rho X + Y\rho Y + Z\rho Z)$ and remembering that $X^2 = Y^2 = Z^2 = \mathbb{1}$, $XY = -YX = Z$, $YZ = -ZY = X$ and $ZX = -XZ = Y$, you can easily verify that $\mathbb{1} = A$ by direct calculation. A nicer way of doing so is, for instance, to show that

$$[A, X] = [A, Y] = [A, Z] = 0, \qquad \text{and} \qquad \mathrm{tr}[A] = 2. \tag{S.1}$$

This proves the claim because the Pauli matrices form a basis of the complex Hilbert space of 2×2 matrices and only multiples of the identity commute with all operators.

From this follows that we can write

$$\mathcal{E}_p(\rho) = p\frac{\mathbb{1}}{2} + (1-p)\rho = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right). \tag{S.2}$$

From that we read out the operator sum representation $\{E_k\}_k$,

$$E_1 = \sqrt{1 - \frac{3p}{4}}\,\mathbb{1}, \quad E_2 = \frac{\sqrt{p}}{2}X, \quad E_3 = \frac{\sqrt{p}}{2}Y, \quad E_4 = \frac{\sqrt{p}}{2}Z. \tag{S.3}$$

(b) Using the result from part (a) we have

$$\mathcal{E}(\rho) = \frac{p}{2}\mathbb{1} + (1-p)\ \rho \tag{S.4}$$

$$= \frac{1}{2}(\mathbb{1} + (1-p)\ \vec{r}\cdot\vec{\sigma}) \tag{S.5}$$

Thus, points on a sphere with radius r are mapped to a smaller sphere with radius $(1-p)r$ — they get more mixed in that sense. In particular, pure states will not remain pure during this TPCPM.

(c) Applying the TPCPM to this state results in

$$\mathcal{E}(\hat{\rho}) = \left(\frac{p}{2} + (1-p)q\right)\ |0\rangle\langle 0|_B + \left(\frac{p}{2} + (1-p)(1-q)\right)\ |1\rangle\langle 1|_B \tag{S.6}$$

The probabilities P_B can be directly read out of the above equation. The conditional probabilities $P_{B|A}$ can be arranged in a transition matrix $(T)_{ij} = P_{B|A}(i|j)$ as follows:

$$T = \begin{pmatrix} \frac{p}{2} + (1-p) & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} + (1-p) \end{pmatrix} = \begin{pmatrix} 1-\frac{p}{2} & \frac{p}{2} \\ \frac{p}{2} & 1-\frac{p}{2} \end{pmatrix}. \tag{S.7}$$

This is what is known as a binary symmetric channel in classical information theory.

Exercise 2. *A sufficient entanglement criterion*

Given a bipartite quantum state ρ_{AB} we say it is separable *if it can be written in the form*

$$\rho_{AB} = \sum_k p_k\, \sigma_A^{(k)} \otimes \sigma_B^{(k)}, \tag{3}$$

where $\{p_k\}_k$ is a probability distribution and $\{\sigma_A^{(k)}\}_k$ and $\{\sigma_B^{(k)}\}_k$ are some states on A and B, respectively. Bipartite states that are not separable are called entangled.

In general it is very difficult to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that correctly identifies all entangled states in low dimensions.

(a) Let $\mathcal{E}_A : \mathrm{End}(\mathcal{H}_A) \to \mathrm{End}(\mathcal{H}_A)$ be a positive superoperator. Show that $\mathcal{E}_A \otimes \mathcal{I}_B$ maps separable states to positive operators.

(b) Let $\{|v_i\rangle_A\}$ be an orthonormal basis for system A and define the transpose $\mathcal{T}$ as

$$\mathcal{T} :\ S = \sum_{ij} s_{ij}\, |v_i\rangle\langle v_j| \mapsto S^T := \sum_{ij} s_{ij}\, |v_j\rangle\langle v_i|. \tag{4}$$

Show that the transpose $\mathcal{T}$ is a positive superoperator and that it is basis dependent.

(c) Define the Werner state on a two-qubit system AB to be

$$W = x\, |\psi^-\rangle\langle\psi^-|_{AB} + (1-x)\, \frac{\mathbb{1}_{AB}}{4}, \tag{5}$$

where $0 \le x \le 1$ and $|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB})$. What happens to the eigenvalues of W if we apply the partial transpose on A to it, i.e., what are the eigenvalues of $W^{T_A} := (\mathcal{T}_A \otimes \mathcal{I}_B)(W)$?

(d) Given a description of a bipartite quantum state, explain how the partial transpose could be used to determine if a state is entangled.

Solution.

(a) Using linearity of the superoperator and applying $\mathcal{E}_A \otimes \mathcal{I}_B$ to each state in the mixture defining the separable state results in a valid product state since $\mathcal{E}_A$ is positive. Mixtures of positive states are positive, so the output is positive.

(b) Since a matrix representation of an operator is in general basis-dependent, so is its transpose.

The positivity of $\mathcal{T}$ follows from

$$\mathcal{T}\left(USU^\dagger\right) = U^{\dagger T} S^T U^T, \tag{S.8}$$

as can be seen from the definition. This means that if S has only positive eigenvalues, so does S^T, because unitaries do not change the eigenvalues, and the transpose of a unitary is unitary.

(c) Clearly the identity remains unchanged by the operation. We write the partial transpose of W in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and obtain

$$W^{T_A} = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & 1-x & -2x & 0 \\ 0 & -2x & 1-x & 0 \\ 0 & 0 & 0 & 1+x \end{pmatrix}. \tag{S.9}$$

There is a doubly-degenerate eigenvalue $(1+x)/4$ and for the other two eigenvalues of W^{T_A} we only need to look at the 2×2 matrix

$$\frac{1}{4}\begin{pmatrix} 1-x & -2x \\ -2x & 1-x \end{pmatrix}.$$

Its eigenvectors are $\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$, with eigenvalues $(1+x)/4$ and $(1-3x)/4$, respectively. The latter is negative for $x > 1/3$, and therefore we can conclude that the state is certainly not separable for $1/3 < x \leq 1$. For this range of x the state is entangled.

Remark 1: The Werner state is an example of an entangled state that nevertheless does not violate Bell's inequality. This means that in this sense the criterion we have constructed is stronger than Bell's inequality.

Remark 2: Indeed, it can be shown that the PPT criterion (positive partial transpose) is necessary and sufficient for bipartite systems of dimension 2×2 and 2×3. Therefore W is only separable for $1/3 > x$.

(d) If the partial transpose applied to A or B of a bipartite state ρ_{AB} has at least one negative eigenvalue, then ρ_{AB} cannot be separable, i.e. has to be entangled. If the partial transpose of ρ_{AB} is positive nothing can be said about whether the state is entangled or not (unless in dimensions 2×2 or 2×3, as mentioned above).