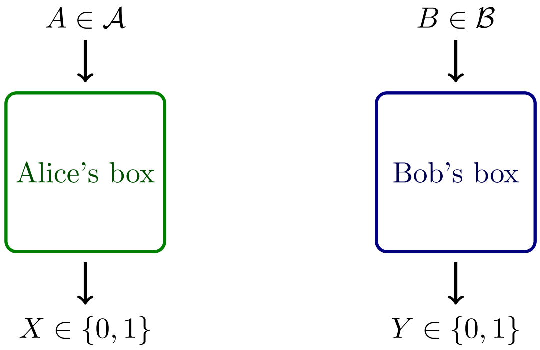
Exercise 1. ***Chained Bell inequalities***

In this exercise we will encounter a Bell violation that is stronger in quantum mechanics than what we have seen so far. Let A and B denote random variables describing the input Alice and Bob give to their devices in space-like separated locations, respectively. The outputs of their devices, described by RVs X and Y, can take on values in $\{0, 1\}$. Alice and Bob can choose their inputs from N different values, $A \in \mathcal{A} = \{0, 2, 4, \ldots, 2N-2\}$ and $B \in \mathcal{B} = \{1, 3, 5 \ldots, 2N-1\}$.



We define I_N, a measure of correlations, by

$$I_N = P[X = Y \mid A = 0, B = 2N-1] \; + \sum_{|a-b|=1} P[X \neq Y \mid A = a, B = b]\,. \tag{1}$$

If I_N is small this implies that the outcomes of adjacent inputs are almost perfectly correlated – a fact that can be used for secret key agreement.

(a) Assuming that the boxes allow for a hidden variable model s.t. X and Y can be seen as independent random variables, show that $I_N \geq 1$.
Hint: Define X_a to be Alice's outcome when she inputs a and Y_b to be Bob's outcome when he inputs b and consider the quantity

$$F_N = 1 - \delta_{X_0 Y_{2N-1}} + \sum_{|a-b|=1} \delta_{X_a Y_b}\,, \tag{2}$$

δ_{xy} being the Kronecker-Delta. Show that for any realisation of the different random variables $F_N \geq 1$ and follow that $I_N \geq 1$.

(b) Within quantum mechanics, e.g. if the boxes contain quantum spins and A and B are inputs defining the measurement basis, one can show that $I_N < 1$ is possible. To see this, assume that Alice and Bob share the 2-qubit state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and perform their measurement in the basis $\{|\frac{k\pi}{2N}\rangle, |\frac{k\pi}{2N} + \pi\rangle\}$ for $k \in \{0, 1, 2, \ldots, 2N-1\}$ (for Alice $k \in \mathcal{A}$, for Bob $k \in \mathcal{B}$). Here, $|\theta\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$.
Show that in this case

$$I_N = 2N \sin^2 \frac{\pi}{4N} \leq \frac{\pi^2}{8N}\,. \tag{3}$$

(c) Consider the case $N = 2$ and compare the above quantum violation of $I_2 \geq 1$ with the violation of the standard Bell inequality.