Exercise 1. ***Trace distance and fidelity: Fuchs-van de Graaf inequalities***

Trace distance $\delta(\rho,\sigma)$ and fidelity $F(\rho,\sigma)$ of two quantum states $\rho,\sigma \in \mathcal{S}(\mathcal{H})$ are closely related. In some sense they can be considered equivalent measures of distance, as we will explore in this exercise. Before we start, let us repeat the quite different definitions of the two objects, δ and F.

$$\begin{aligned}
\delta(\rho,\sigma) &:= \operatorname{tr}|\rho-\sigma| \equiv \operatorname{tr}\left[\sqrt{(\rho-\sigma)^\dagger(\rho-\sigma)}\right] \\
&= \max_{P \text{ proj.}} \operatorname{tr}\left[P(\rho-\sigma)\right] && \text{(alternative def.)} \\
F(\rho,\sigma) &:= \operatorname{tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right] \\
&= \max_{|\Psi\rangle,|\Phi\rangle \text{ purif.}} |\langle\Psi|\Phi\rangle| && \text{(Uhlmann)}
\end{aligned}$$

(a) Show that in the case of pure states $\rho = |\psi\rangle\langle\psi|,\ \sigma = |\phi\rangle\langle\phi|$ trace distance and fidelity fulfil

$$\delta(\rho,\sigma) = \sqrt{1-F(\rho,\sigma)^2}\,. \tag{1}$$

(b) Use that trace distance can only decrease under quantum operations (see last sheet) to show that for general $\rho,\sigma \in \mathcal{S}(\mathcal{H})$

$$\delta(\rho,\sigma) \leq \sqrt{1-F(\rho,\sigma)^2}\,. \tag{2}$$

There is yet another very useful characterization of the fidelity as an optimization over all possible POVM measurements. For two (classical) probability distributions $\{p_m\}_m$ and $\{q_m\}_m$ define the *classical* fidelity to be

$$F(\{p_m\},\{q_m\}) := \sum_m \sqrt{p_m q_m}\,.$$

The *quantum* fidelity can then be written as

$$F(\rho,\sigma) = \min_{\{E_m\} \text{ POVM}} F(\{p_m\},\{q_m\}), \tag{3}$$

where $p_m := \operatorname{tr}[\rho\, E_m]$ and $q_m := \operatorname{tr}[\sigma\, E_m]$. Likewise, using the same notation, the *quantum* trace distance can be written as

$$\delta(\rho,\sigma) = \max_{\{E_m\} \text{ POVM}} \delta(\{p_m\},\{q_m\}), \tag{4}$$

where $\delta(\{p_m\},\{q_m\})$ is the *classical* trace distance of the respective probability distributions.

(c) Use this way of writing $F(\rho,\sigma)$ and $\delta(\rho,\sigma)$ to prove that for any two states $\rho,\sigma \in \mathcal{S}(\mathcal{H})$

$$1-F(\rho,\sigma) \leq \delta(\rho,\sigma)\,. \tag{5}$$

In total this shows 'equivalence' of δ and F in terms of the inequalities

$$1-F(\rho,\sigma) \leq \delta(\rho,\sigma) \leq \sqrt{1-F(\rho,\sigma)^2}\,.$$

Exercise 2. ***Properties of von Neumann entropy***

The *von Neumann entropy* of a density operator $\rho \in \mathcal{S}(\mathcal{H}_A)$ is defined as $H(A)_\rho := -\mathrm{tr}\big(\rho \log \rho\big)$. Given a composite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ we write $H(AB)_\rho$ to denote the von Neumann entropy of the reduced state of a subsystem, $\rho_{AB} = \mathrm{tr}_C(\rho_{ABC})$. When the state ρ is obvious from the context we can drop the index.
The *conditional* von Neumann entropy may be defined as $H(A|B)_\rho := H(AB)_\rho - H(B)_\rho$. In the Alice-and-Bob picture this quantifies the uncertainty that Bob, who holds part of a quantum state, ρ_B, still has about Alice's state.
The *strong sub-additivity* property of the von Neumann entropy shows up a lot. It applies to a tripartite composite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$,

$$H(A|BC)_\rho \leq H(A|B)_\rho. \tag{6}$$

(a) Prove the following general properties of the von Neumann entropy.

  (i) If ρ_{AB} is pure, then $H(A)_\rho = H(B)_\rho$.
  (ii) If ρ_{ABC} is pure, then $H(A|C)_\rho = -H(A|B)_\rho$.
  (iii) If two systems are independent, $\rho_{AB} = \rho_A \otimes \rho_B$, then $H(AB)_\rho = H(A)_{\rho_A} + H(B)_{\rho_B}$.

(b) Consider a bipartite state that is classical on subsystem Z: $\rho_{ZA} = \sum_z p_z |z\rangle\langle z|_Z \otimes \rho_A^z$ for some orthogonal basis $\{|z\rangle_Z\}_z$ of $\mathcal{H}_Z$ and a set of states $\{\rho_A^z\}_z \subset \mathcal{S}(\mathcal{H}_A)$. Show that:

  (i) The conditional entropy of the quantum part, A, given the classical information Z is

$$H(A|Z)_\rho = \sum_z p_z \, H(A|Z = z), \tag{7}$$

  where $H(A|Z = z) = H(A)_{\rho_A^z}$.
  (ii) The entropy of A is concave,

$$H(A)_\rho \geq \sum_z p_z \, H(A|Z = z). \tag{8}$$

  (iii) The entropy of a classical probability distribution $\{p_z\}_z$ cannot be negative, even if one has access to extra quantum information, A,

$$H(Z|A)_\rho \geq 0. \tag{9}$$

  Remark: Eq (9) holds in general only for classical Z. Bell states are immediate counterexamples in the fully quantum case.

Exercise 3. ***Upper bound on von Neumann entropy***

(a) Given a state $\rho \in \mathcal{S}(\mathcal{H}_A)$, show that

$$H(A)_\rho \leq \log |\mathcal{H}_A| \,. \tag{10}$$

Hints: Consider the state $\bar{\rho} = \int U \rho U^\dagger dU$, where the integral is over all unitaries $U \in \mathcal{U}(\mathcal{H}_A)$ and dU is the Haar measure. Find $\bar{\rho}$ and use concavity, (8), to show (10). The Haar measure satisfies $d(UV) = d(VU) = dU$, where $V \in \mathcal{U}(\mathcal{H}_A)$ is any fixed unitary.

(b) For $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, show that the conditional entropy satisfies

$$-\log |\mathcal{H}_A| \leq H(A|B)_\rho \leq \log |\mathcal{H}_A| \,. \tag{11}$$