Exercise 1. ***Depolarizing channel***

We are given two two-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and a completely positive trace preserving (CPTP) map $\mathcal{E}_p : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$, $0 \leq p \leq 1$, defined as

$$\mathcal{E}_p(\rho) = p\frac{\mathbb{1}}{2} + (1-p)\rho. \tag{1}$$

(a) An operator-sum representation (also called the Kraus-operator representation) of a CPTP map $\mathcal{E} : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ is a decomposition $\{E_k\}_k$ of operators $E_k \in \mathrm{Hom}(\mathcal{H}_A, \mathcal{H}_B)$, $\sum_k E_k {E_k}^\dagger = \mathbb{1}$, such that

$$\mathcal{E}(\rho) = \sum_k E_k \rho {E_k}^\dagger.$$

Find an operator-sum representation for $\mathcal{E}_p$.

Hint: Remember that $\rho \in \mathcal{S}(\mathcal{H}_A)$ can be written in the Bloch sphere representation:

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}), \quad \vec{r} \in \mathbb{R}^3, \quad |\vec{r}| \leq 1, \quad \vec{r}\cdot\vec{\sigma} = r_x\sigma_x + r_y\sigma_y + r_z\sigma_z, \tag{2}$$

where σ_x, σ_y and σ_z are Pauli matrices. It may be useful to show that

$$\mathbb{1} = \frac{1}{2}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z).$$

(b) What happens to the radius $\vec{r}$ when we apply $\mathcal{E}_p$? How can this be interpreted?

(c) A probability distribution $P_A(0) = q$, $P_A(1) = 1 - q$ can be encoded in a quantum state on $\mathcal{H}_A$ as $\hat{\rho} = q|0\rangle\langle 0|_A + (1-q)|1\rangle\langle 1|_A$. Calculate $\mathcal{E}(\hat{\rho})$ and the conditional probabilities $P_{B|A}$ as well as P_B after measuring $\mathcal{E}(\hat{\rho})$ in the standard basis $\{|0\rangle_B, |1\rangle_B\}$.

Exercise 2. ***A sufficient entanglement criterion***

Given a bipartite quantum state ρ_{AB} we say it is *separable* if it can be written in the form

$$\rho_{AB} = \sum_k p_k\, \sigma_A^{(k)} \otimes \sigma_B^{(k)}, \tag{3}$$

where $\{p_k\}_k$ is a probability distribution and $\{\sigma_A^{(k)}\}_k$ and $\{\sigma_B^{(k)}\}_k$ are some states on A and B, respectively. Bipartite states that are not separable are called *entangled.*

In general it is very difficult to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that correctly identifies all entangled states in low dimensions.

(a) Let $\mathcal{E}_A : \mathrm{End}(\mathcal{H}_A) \to \mathrm{End}(\mathcal{H}_A)$ be a positive superoperator. Show that $\mathcal{E}_A \otimes \mathcal{I}_B$ maps separable states to positive operators.

(b) Let $\{|v_i\rangle_A\}$ be an orthonormal basis for system A and define the transpose $\mathcal{T}$ as

$$\mathcal{T}:\ S = \sum_{ij} s_{ij}\, |v_i\rangle\langle v_j| \mapsto S^T := \sum_{ij} s_{ij}\, |v_j\rangle\langle v_i|. \tag{4}$$

Show that the transpose $\mathcal{T}$ is a positive superoperator and that it is basis dependent.

(c) Define the Werner state on a two-qubit system AB to be

$$W = x\, |\psi^-\rangle\langle\psi^-|_{AB} + (1-x)\, \frac{\mathbb{1}_{AB}}{4}, \tag{5}$$

where $0 \leq x \leq 1$ and $|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB})$. What happens to the eigenvalues of W if we apply the partial transpose on A to it, i.e., what are the eigenvalues of $W^{T_A} := (\mathcal{T}_A \otimes \mathcal{I}_B)(W)$?

(d) Given a description of a bipartite quantum state, explain how the partial transpose could be used to determine if a state is entangled.