Exercise 1. ***Partial trace***

The partial trace is an important concept in the quantum mechanical treatment of multi-partite systems, and it is the natural generalisation of the concept of marginal distributions in classical probability theory. Let ρ_{AB} be a density matrix on the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_A = \mathrm{tr}_B(\rho_{AB})$ the marginal on $\mathcal{H}_A$.

(a) Show that ρ_A is a valid density operator by proving it is:

  (i) Hermitian: $\rho_A = \rho_A^\dagger$.
  (ii) Positive: $\rho_A \geq 0$.
  (iii) Normalised: $\mathrm{tr}(\rho_A) = 1$.

(b) Calculate the reduced density matrix of system A in the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right), \quad \text{where} \quad |ab\rangle = |a\rangle_{\mathrm{A}} \otimes |b\rangle_{\mathrm{B}}. \tag{1}$$

(c) Consider a classical probability distribution P_{XY} with marginals P_X and P_Y.

  (i) Calculate the marginal distribution P_X for

$$P_{XY}(x,y) = \begin{cases} 0.5 & \text{for } (x,y) = (0,0), \\ 0.5 & \text{for } (x,y) = (1,1), \\ 0 & \text{else,} \end{cases} \tag{2}$$

  with alphabets $\mathcal{X}, \mathcal{Y} = \{0,1\}$.

  (ii) How can we represent P_{XY} in form of a quantum state?
  (iii) Calculate the partial trace of P_{XY} in its quantum representation.

(d) Can you think of an experiment to distinguish the bipartite states of parts (b) and (c)?

Exercise 2. ***Bipartite systems and measurement***

(a) Consider a state ρ_{AB} in a composed system $\mathcal{H}_A \otimes \mathcal{H}_B$ shared by Alice, who is in possession of system A, and Bob, who has access to B. Suppose Alice wants to perform a measurement described by an observable O_A on subsystem $\mathcal{H}_A$. The operator O_A has eigenvalues (possible outcomes) $\{x\}_x$ and may be written as the spectral decomposition $O_A = \sum_x x\, P_x$, where $\{P_x\}_x$ are projectors – operators that only have eigenvalues 0 and 1.

Show that the measurement statistics (probabilities of obtaining the different outcomes) are the same whether you apply $O_A \otimes \mathbb{1}_B$ on the joint state ρ_{AB} or first trace out the system $\mathcal{H}_B$ and then apply O_A on the reduced state ρ_A.

(b) Suppose now that Alice and Bob share a two-qubit system in a maximally entangled state,

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \right). \tag{3}$$

Alice then performs a measurement in the basis $\{|\theta\rangle, |\frac{\pi}{2}+\theta\rangle\}$, with $|\theta\rangle := \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$ for $\theta \in \mathbb{R}$, on her qubit. In the notation from (a) the basis corresponds to the projectors $P_{+1} = |\theta\rangle\langle\theta|$ and $P_{-1} = |\frac{\pi}{2}+\theta\rangle\langle\frac{\pi}{2}+\theta|$.

(i) What description does Alice give to system B, given the outcome of her measurement?

(ii) If Bob performs the measurement in the basis $\{|0\rangle, |1\rangle\}$ on his part of the system, B, what is the probability distribution for his outcomes? How would Alice, who knows the outsome of her measurement, describe his probability distribution?

(c) Finally, Alice and Bob share an arbitrary pure state $|\Psi\rangle_{AB}$, and Bob would like to perform a measurement on B described by projectors $\{Q_y\}_y$. Unfortunately his measurement apparatus is broken, however he can still perform arbitrary unitary operations. Meanwhile, Alice's measurement apparatus is in good working order. Show that there exist projectors $\{P_y\}_y$ on Alice's part and unitaries U_y on A and V_y on B so that

$$|\widetilde{\Psi}_y\rangle_{AB} := (\mathbb{1}_A \otimes Q_y)\,|\Psi\rangle_{AB} = (U_y \otimes V_y)\,(P_y \otimes \mathbb{1}_B)\,|\Psi\rangle_{AB}. \tag{4}$$

(Note that the 'state' $|\widetilde{\Psi}_y\rangle_{AB}$ is unnormalized, so that it implicitly encodes the probability of outcome y.)

By showing this we have proven that Alice can assist Bob by performing a related measurement herself, after which they can *locally* correct the state using the *local* unitaries U_y and V_y. Notice that Alice will have to (classically) communicate to Bob what her outcome was.

Hint: Use the Schmidt decomposition and work in the Schmidt basis of $|\Psi\rangle_{AB}$.