Exercise 1. ***Gravitational waves***

Consider the following metric for a space-time, parameterised locally by (u^1, u^2, u^3, u^4):

$$ds^2 = g_{\mu\nu}\, du^\mu\, du^\nu = -\,(du^1\, du^2 + du^2\, du^1) + a^2(u^1)\,(du^3)^2 + b^2(u^1)\,(du^4)^2 \,, \tag{1}$$

where a and b are unspecified functions of u^1. For appropriate choices of a and b, this represents an exact gravitational plane wave.

i) Calculate the Christoffel symbols

$$\Gamma^\kappa_{\mu\nu} = \frac{1}{2}\, g^{\kappa\rho}\,(\partial_\mu\, g_{\rho\nu} + \partial_\nu\, g_{\rho\mu} - \partial_\rho\, g_{\mu\nu}) \tag{2}$$

and the Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\nu \Gamma^\rho_{\mu\sigma} - \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\nu\kappa}\,\Gamma^\kappa_{\mu\sigma} - \Gamma^\rho_{\mu\kappa}\,\Gamma^\kappa_{\nu\sigma} \tag{3}$$

for the metric (1).

ii) Use Einstein's equation in vacuum to derive equations obeyed by $a(u^1)$ and $b(u^1)$.

iii) Show that an exact solution can be found, in which both a and b are functionals of an arbitrary function $f(u^1)$.

iv) Repeat the calculation of the curvature tensor for the metric (1) using form language:

a) determine the tetrad 1-forms,

$$e^a = e^a_\mu\, du^\mu \,, \tag{4}$$

where

$$\eta_{ab}\, e^a_\mu\, e^b_\nu = g_{\mu\nu} \,, \qquad \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1) \,. \tag{5}$$

b) Use the torsion-free condition,

$$De^a = d\,e^a + \omega^a{}_b \wedge e^b = 0 \,, \tag{6}$$

to show that the (metric compatible) Ricci spin connection,

$$\omega^a{}_b = \omega_\mu{}^a{}_b\, du^\mu \tag{7}$$

can be determined by the following formula:

$$\omega_\mu{}^a{}_b = \frac{1}{2}\, e^d_\mu\, \eta^{ac} \Big(\langle [e_d, e_b], e_c \rangle + \langle [e_c, e_d], e_b \rangle - \langle [e_b, e_c], e_d \rangle \Big) \,, \tag{8}$$

where the $e_a = e^\mu_a \partial_\mu$ are vector fields associated to the inverse tetrad $e^\mu_a = g^{\mu\nu}\, \eta_{ab}\, e^b_\nu$, and

$$\langle e_a, e_b \rangle = g_{\mu\nu}\, e^\mu_a\, e^\nu_b \tag{9}$$

is the inner product with respect to the metric $g_{\mu\nu}$. [Hint: apply the two-form (6) to the bi-vector $e_c \otimes e_d$, i.e., $De^a(e_c, e_d) = 0$ and lower temporarily the index a by contracting it with η_{ae}. Then solve the system of equations obtained by permuting the indices d, b, c for ω_{dbc}. Note that ω_{dbc} is anti-symmetric in the indices b, c (why?).]

c) Use (8) to determine the spin connection for the tetrad field from part a).

d) Compute the curvature 2-form

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \,. \tag{10}$$

Exercise 2. ***Gravitational wave detection***

Gravitational waves can be detected by monitoring the distances between two freely flying masses. If one of the masses is equipped with a laser and an accurate clock, and the other with a good mirror, the distance between the masses can be measured by timing how long it takes for a pulse of laser light to make the round-trip journey.

i) How would you want your detector oriented to register the largest response from a plane wave of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + [1 + A\cos(\Omega\,(t - z))]\,\mathrm{d}x^2 + [1 - A\cos(\Omega\,(t - z))]\,\mathrm{d}y^2 + \mathrm{d}z^2 \,, \quad (11)$$

where Ω is the frequency of the wave? [Note that (11) satisfies the vacuum Einstein equations only up to terms of $\mathcal{O}(A^2)$.]

ii) If the masses have a mean separation L, what is the largest change in the arrival time of the pulses caused by the wave?

iii) What frequencies Ω would go undetected?

[Hint: work in linearised gravity. We already have a metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (12)$$

with h small. Decompose also the geodesics of the light-rays as

$$x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda)\,, \quad (13)$$

and write

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^{(0)\mu}}{\mathrm{d}\lambda} + \frac{\mathrm{d}x^{(1)\mu}}{\mathrm{d}\lambda} = k^\mu + l^\mu, \quad (14)$$

where k^μ is the 4-velocity of the background geodesic and l^μ is a perturbation. Now solve the geodesic equation separately at zero'th and first order in h and l to obtain a formula for the change in $x^{(1)0}$ along the round-trip.]

Exercise 3. ***Binary star***

Consider a binary system consisting of two stars of equal mass M. We assume that the size of the stars is negligible, and that the stars move on a nearly Newtonian circular orbit of radius R around each other (i.e., the distance between the stars is $2\,R$). Due to gravitational wave radiation, the system looses energy at a rate of

$$P = -\frac{G}{5}\sum_{i,j=1}^{3}\left(\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\right)^2 , \quad (15)$$

where

$$Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij} q \quad (16)$$

with $q = \sum_{i=1}^{3} q_{ii}$ and

$$q_{ij} = 3\int T^{00} x^i x^j \mathrm{d}^3 x\,. \quad (17)$$

[Here T^{00} is the energy density of the system in the center of mass rest frame.]

i) Calculate the (initial) angular frequency of the two stars and parametrise their motion in Minkowski space (assume they move in the $x^1 - x^2$ plane).

ii) What is the energy density T^{00} for this binary star system?

iii) Using equations (15), (16) and (17), calculate the energy loss due to gravitational radiation.

iv) Calculate the rate of increase of the orbital frequency due to the emission of gravitational waves.