Exercise 1. *Linearised gravity*

i) In linearised gravity, we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We change coordinates by defining

$$\tilde{x}^\mu = x^\mu - \epsilon\xi^\mu(x) + \mathcal{O}(\epsilon^2) \ , \tag{1}$$

where $\xi^\mu(x)$ is a vector field. Show that under this change of coordinates, $h_{\mu\nu}$ changes to

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \epsilon(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) + \mathcal{O}(\epsilon^2) \ . \tag{2}$$

ii) Calculate the Ricci tensor corresponding to $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ to second order in $h_{\mu\nu}$, starting from the definition of the Ricci tensor in terms of the metric and the Christoffel symbols,

$$R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu} = \partial_\nu\Gamma^\nu_{\mu\rho} - \partial_\mu\Gamma^\nu_{\nu\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\nu_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\nu_{\alpha\mu} \ . \tag{3}$$

Exercise 2. *Geodesics on a 2-sphere*

Show that geodesics on a 2-sphere are great circles. Compare the geodesic equations to the equations of motion (obtained from the Euler Lagrange equations) of the spherical pendulum in the absence of gravity.

Hint: A spherical pendulum consists of a mass m attached to a massless rigid string of length l. The Lagrangian (including gravity) is given by $L = T - V$, where $T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$ and $V = -mgl\cos\theta$.

Exercise 3. *Isometries*

Show that an isometry implies a conserved charge for the dynamics of a particle, the conserved charge (along geodesics) being $q = g_{\mu\nu}\dot{x}^\mu\xi^\nu$ where ξ^ν is the Killing vector. Show this (i) by showing that $\frac{d}{d\lambda}q = 0$ where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ and, (ii) using Noether's theorem.

Hint: An infinitesimal transformation $x^\mu \to x^\mu + \delta x^\mu$ is a symmetry if the corresponding off-shell variation of the Lagrangian is at most a total derivative, i.e., $\delta L = \frac{d\theta}{d\lambda}$ for some infinitesimal function $\theta(x^\mu, \dot{x}^\mu)$ (*without using the equations of motion*). Noether's theorem then implies that the charge $q = \left(\frac{dL}{d\dot{x}^\mu}\delta x^\mu - \theta\right)$ is conserved when the equations of motion hold, i.e., $\frac{dq}{d\lambda} = 0$ on-shell.