Exercise 1. ***Symmetries of Riemann and Weyl tensor***

The Riemann tensor $R_{\mu\nu\rho\lambda}$ has the properties

$$R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda} \ , \qquad R_{[\mu\nu\rho]\lambda} = 0 \ , \qquad R_{\mu\nu\rho\lambda} = -R_{\mu\nu\lambda\rho} \ . \tag{1}$$

i) Show that it satisfies

$$R_{\mu\nu\rho\lambda} = R_{\rho\lambda\mu\nu} \ .$$

ii) The Ricci tensor is defined by

$$R_{\mu\nu} = R_{\mu\rho\nu\lambda}\, g^{\rho\lambda} \ .$$

Show that the Ricci tensor is symmetric,

$$R_{\mu\nu} = R_{\nu\mu} \ .$$

iii) For $n > 2$, where n is the dimension of the manifold, we define the Weyl tensor $C_{\mu\nu\rho\lambda}$ by the equation

$$R_{\mu\nu\rho\lambda} = C_{\mu\nu\rho\lambda} + \frac{2}{n-2}\Big(g_{\mu[\rho}R_{\lambda]\nu} - g_{\nu[\rho}R_{\lambda]\mu}\Big) - \frac{2}{(n-1)(n-2)} R g_{\mu[\rho}g_{\lambda]\nu} \ ,$$

where R is the scalar curvature defined by

$$R = R_{\mu\nu}g^{\mu\nu} \ .$$

Show that the Weyl tensor has the same symmetry properties as the Riemann tensor, *i.e.*

$$C_{\mu\nu\rho\lambda} = -C_{\nu\mu\rho\lambda} \ , \qquad C_{[\mu\nu\rho]\lambda} = 0 \ , \qquad C_{\mu\nu\rho\lambda} = -C_{\mu\nu\lambda\rho} \ .$$

Furthermore, show that the Weyl tensor is traceless with respect to the contraction of any pair of indices.

Exercise 2. ***Metric and Riemann Tensor of 2-sphere***

i) Show that in two dimensions, the Riemann tensor takes the form

$$R_{abcd} = R g_{a[c}g_{d]b} \ . \tag{2}$$

(Hint: First deduce that the Riemann tensor has only one independent component in two dimensions. Then show that $g_{a[c}g_{d]b}$ spans the vector space of tensors having the symmetries of the Riemann tensor.)

ii) Determine the metric on the surface of a sphere of radius r in the usual spherical coordinates (θ, ϕ). Determine also the inverse metric $g^{\alpha\beta}$.

iii) Calculate the Riemann curvature tensor of the sphere. (Hint: The Riemann tensor in terms of the Christoffel symbols is given by

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \ .$$

Because of (i), there is only one independent component which you can take to be $R_{\theta\phi\theta\phi}$. Determine all other components in terms of it.)

Exercise 3. ***Affine Parametrisation of Curves***

i) A geodesic $\gamma(t)$ is characterised by the property that the tangent vector is parallely propagated along itself, i.e., that the tangent vector $T = \frac{d\gamma(t)}{dt}$ satisfies

$$T^a \nabla_a T^b = \alpha T^b \ , \tag{3}$$

where α is some constant. Show that one can always find a parametrisation of the curve $t \equiv t(s)$ so that (3) becomes

$$S^a \nabla_a S^b = 0 \ ,$$

where S is the tangent vector with respect to s. The resulting parametrisation is called the affine parametrisation.
(Hint: Work in coordinates.)

ii) Let t be an affine parameter of a geodesic γ. Show that any other affine parameter s of γ takes the form $s = at + b$, where a and b are constants.

iii) Let $\gamma_s(t)$ be a smooth one-parameter family of geodesics, i.e., for each $s \in \mathbb{R}$, $\gamma_s(t)$ is a geodesic parametrised by an affine parameter t. The vector field $X = \frac{\partial}{\partial s}$ represents the displacement of nearby geodesics and is called the deviation vector. Because of (ii) there is a 'gauge freedom' in the definition of X since we can change the t-parameterisations in an s-dependent manner, i.e.,

$$t \mapsto t' = a(s)t + b(s) \ .$$

Show that this modifies X by adding to it a multiple of $T = \frac{\partial}{\partial t}$. For the case where the geodesics are timelike or spacelike show that we can use this gauge freedom to choose X^a always orthogonal to T^b, i.e.,

$$g_{ab} X^a T^b = 0 \ .$$

(Hint: For the last claim, use the geodesic deviation equation

$$T^\mu \nabla_\mu (T^\nu \nabla_\nu X^\lambda) = -R_{\mu\nu\rho}{}^\lambda X^\nu T^\mu T^\rho \ .)$$