9.1. Compactification on T^2 with constanti Kalb-Ramond field

Consider a closed string theory. Assume that the directions x^2 and x^3 are each compactified onto a circle of radius R; the corresponding string coordinated are called X^r, $r = 2, 3$. Consider a configuration with nonvanishing Kalb-Ramond field with expectation value

$$B_{23} \equiv \frac{1}{2\pi\alpha'} b\,, \qquad \text{all other components vanish.} \tag{9.1}$$

a) Build an action for the $X^r(\tau, \sigma)$ by adding the term

$$S = -\int \mathrm{d}\tau \mathrm{d}\sigma \, \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\mu}{\partial \sigma} B_{\mu\nu} \tag{9.2}$$

to the usual kinetic term

$$S_{\text{kin}} = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, (\dot{X}^2 - X'^2)\,. \tag{9.3}$$

For closed strings we implicitly choose the range of σ to be $\sigma \in [0, 2\pi]$.

b) Consider the mode expansion for the zero-mode part of the compactified coordinate fields

$$X^r(\tau, \sigma) = x^r(\tau) + m_r R \sigma\,, \qquad r = 2, 3\,. \tag{9.4}$$

Compute the Lagrangian (*not* the Lagrangian density!) for the dynamical variables $x^r(\tau)$. *Hint:* you should get a total derivative term; do not discard it, as it is important for the quantum theory!

c) Compute the Hamiltonian from the Lagrangian previously computed. Verify that the Hamiltonian generates the correct equations of motion.

d) Until now, we have only looked at the zero modes. However, the oscillator expansion of the coordinates works as before. Write the appopriate expansion for the compactified coordinates X^r along the lines of

$$X(\tau, \sigma) = x_0 + \alpha' p \, \tau + \alpha' \, w \, \sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (\tilde{\alpha}_n \, e^{-in\sigma} + \alpha_n e^{in\sigma})\,. \tag{9.5}$$

Compute the mass-squared operator (where here we mean the operator $-p^2$ along the noncompact directions) and show that it takes the form (recall that the momenta p_r are quantized in units of $\frac{1}{R}$)

$$M^2 = \left(\frac{n_2}{R} + b\frac{m_3 R}{\alpha'}\right)^2 + \left(\frac{n_3}{R} - b\frac{m_2 R}{\alpha'}\right)^2 + \left(\frac{m_2 R}{\alpha'}\right)^2 + \left(\frac{m_3 R}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2)\,. \tag{9.6}$$

Hint: recall that $M^2 = 2p^+p^- - p^i p^i$ where again i runs over noncompact transverse directions. Moreover, recall that

$$L_0^\perp = \frac{1}{2}\alpha_0^I \alpha_0^I + N^\perp\,, \qquad \bar{L}_0^\perp = \frac{1}{2}\tilde{\alpha}_0^I \tilde{\alpha}_0^I + \bar{N}^\perp\,. \tag{9.7}$$

Split the transverse I index into compact and noncompact contributions and express them in terms of p^i and α_0^r, $\tilde{\alpha}_0^r$. Finally, recall that

$$\alpha' p^+ p^- = L_0^\perp + \bar{L}_0^\perp - 2\,. \tag{9.8}$$

9.2. Kaluza-Klein theory

Already as early as 1921 T. Kaluza proposed a model to unify gravity and electromagnetism by introducing a (large) fifth dimension. O. Klein greatly improved this ansatz by introducing the idea of a compact fifth dimension. In Kaluza-Klein theory the metric tensor G_{MN} in $4+1$ dimensions is split into 3 fields: a metric field $g_{\mu\nu}$ in $3+1$, a $U(1)$ gauge field A_μ and a scalar *dilaton* ϕ in the following way

$$G_{MN} = \phi^{-\frac{1}{3}} \begin{pmatrix} (g_{\mu\nu} + \kappa^2 \phi A_\mu A_\nu) & +\kappa\phi A_\mu \\ +\kappa\phi A_\nu & \phi \end{pmatrix}\,. \tag{9.9}$$

a) We begin by considering a massless scalar field Φ in D dimensions. Argue that the compactification of one spacelike dimension on a circle of radius R as

$$x^{D-1} \sim x^{D-1} + 2\pi R \tag{9.10}$$

will give you an infinite tower of massive scalar fields in $D-1$ dimensions and one massless scalar. *Hint:* Contemplate the kind of momenta you get in compact spaces. Use the Fourier series. Write down the equations of motion of the scalar in D dimensions to see which $(D-1)$-dimensional modes are massive and which massless.

b) How can you make this tower of massive scalar fields vanish or at least impossible to detect at small energies?

c) Now we return to KK theory. $5D$ gravity is of course invariant under reparametrisations. However after singling out the fifth dimension we can only have $4D$ reparametrisations and a transformation

$$x^\mu = \tilde{x}^\mu\,, \qquad \text{and} \qquad x^4 + f(x^\mu) = \tilde{x}^4\,. \tag{9.11}$$

Show that this transformation implies a gauge transformation on the field A_μ.

d) [*Advanced*] To proceed, Klein set $\phi = 1$ and demanded that all equations for the lower dimensional fields follow from the five-dimensionad vacuum Einstein equations

$$R_{MN} = 0\,. \tag{9.12}$$

Derive the equations of motion $R_{D-1,D-1} = 0$, by setting derivatives w.r.t. the fifth dimension of the fields to zero and show that Klein's assumption leads to an unphysical contraint on the gauge field. *Hint;* recall that the Ricci tensor is the trace of the Riemann tensor over the first and third index, $R_{MN} := R^P{}_{M,PN}$.