5.1. Light-cone gauge and motion of open strings

Consider the string motion described by

$$\begin{aligned} X^0(\tau,\sigma) &= \sqrt{2}\,a\left(\tau + \frac{1}{4}\sin 2\tau\,\cos 2\sigma\right), \\ X^1(\tau,\sigma) &= -\frac{a}{2\sqrt{2}}\sin 2\tau\,\cos 2\sigma\,, \\ X^2(\tau,\sigma) &= 2a\,\sin\tau\,\cos\sigma\,, \end{aligned} \tag{5.1}$$

where a is a dimensionless constant and we set $\alpha' = \frac{1}{2}$.

a) Show that this solution is in light-cone gauge and with the correct σ-parametrization. The lightlike reference vector is $n^\mu = \frac{1}{\sqrt{2}}(1,1,0)$. Compute p^+.

b) Show that this solution satisfies the constraints $\dot{X}^2 + X'^2 = \dot{X}\cdot X' = 0$.

c) Compute the mode expansion of the solution. *Hint:* the only transverse direction is X^2.

d) What is the spatial length of the string at $t = X^0 = 0$? Try to plot the motion of the string in the x^1, x^2 plane as τ (or t) flows. What kind of motion does the string follow?

5.2. From the Witt algebra to Virasoro algebra

The goal of this exercise is to show that the Virasoro algebra is the only nontrivial central extension of the Witt algebra. Recall that the Witt algebra is the complex Lie algebra spanned by generators $\{\ell_m\}_{m\in\mathbb{Z}}$ satisfying

$$[\ell_m, \ell_m] = (m-n)\ell_{m+n}\,. \tag{5.2}$$

Consider the central extension spanned by $\{L_m\}_{m\in\mathbb{Z}}$, satisfying

$$[L_m, L_n] = (m-n)L_{m+n} + c_{m,n}\,, \tag{5.3}$$

where the $c_{m,n}\in\mathbb{C}$ are in the center of the algebra (meaning that they commute with all elements of the algebra).

a) Use the antisymmetry of the Lie bracket and the Jacobi identity to prove that $c_{m,n}$ is antisymmetric in m and n and satisfies the equation

$$(n-k)c_{m,n+k} + (k-m)c_{n,m+k} + (m-n)c_{k,m+n} = 0\,. \tag{5.4}$$

b) Show that the terms $c_{n,0}, c_{0,n}$ and $c_{1,-1}$ can be set to zero by the redifinition of the generators

$$\begin{aligned} \tilde{L}_n &= L_n + \frac{c_{n,0}}{n}\,, \qquad n\neq 0\,, \\ \tilde{L}_0 &= L_0 + \frac{1}{2}c_{1,-1}\,. \end{aligned} \tag{5.5}$$

c) Use eq. (5.4) to show that $c_{m,n} = 0$ if $m + n \neq 0$.

d) Use eq. (5.4) to obtain a recursion relation for the coefficients $\tilde{c}_m$, where we define $c_{m,n} := \delta_{m,-n}\tilde{c}_m$. How many free parameters does the solution to the recursion relation admit?

e) Use all you have derived so far to solve the recursion relation. What happens if you impose that $L_0, L_{\pm 1}$ span $\mathfrak{sl}(2,\mathbb{C})$?

5.3. Analytic properties of the zeta function

One of the most remarkable formulas in mathematics is the celebrated "identity"

$$1 + 2 + 3 + 4 + 5 + \ldots = -\frac{1}{12} \tag{5.6}$$

In this exercise we will try to understand where the result comes from by analytically continuing the ζ-function. The Γ and ζ functions of a complex variable z are given by

$$\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1} \quad \text{for } \mathrm{Re}(z) > 0 \quad \text{and} \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \text{for } \mathrm{Re}(z) > 1\,. \tag{5.7}$$

a) We start by regularising $\zeta(-1)$. Show that we can write the zeta function as

$$\zeta_\epsilon(-1) = -\frac{\partial}{\partial \epsilon} \sum_{n=1}^{\infty} e^{-n\epsilon}. \tag{5.8}$$

in the limit $\epsilon \to 0$. Argue that the sum in this expression is convergent and give the solution. Expand the expression for small ϵ and show that the result is given by $\zeta_\epsilon(-1) \approx \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon)$.

b) Show that for $\mathrm{Re}(z) > 1$ you can write

$$\Gamma(z)\zeta(z) = \int_0^\infty dt \frac{t^{z-1}}{e^t - 1}. \tag{5.9}$$

Conclude that the analytic continuation of the above expression for $\mathrm{Re}(z) > -2$ is

$$\Gamma(z)\zeta(z) = \int_0^1 dt\, t^{z-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) + \frac{1}{z-1} - \frac{1}{2z} + \frac{1}{12(z+1)} + \int_1^\infty dt \frac{t^{z-1}}{e^t - 1} \tag{5.10}$$

c) We know that $\Gamma(z)$ has simple poles for $z = 0, -1, -2, \ldots$, with residues

$$\mathrm{Res}_{z=-n}\big[\Gamma(z)\big] = \frac{(-1)^n}{n!}\,. \tag{5.11}$$

Conclude that the values of ζ at 0 and -1 are

$$\zeta(0) = -\frac{1}{2} \quad \text{and} \quad \zeta(-1) = -\frac{1}{12}. \tag{5.12}$$