## 3.1. Jumping a relativistic rope

Consider a relativistic open string in static gauge, with fixed endpoints

$$\vec{X}(t,0) \equiv \vec{x}_1, \quad \vec{X}(t,\sigma_1) \equiv \vec{x}_2 \,. \tag{3.1}$$

The boundary condition at $\sigma = 0$ is solved by the following solution to the wave equation

$$\vec{X}(t,\sigma) = \vec{x}_1 + \frac{1}{2}\Big[\vec{F}(ct-\sigma) - \vec{F}(ct+\sigma)\Big] \tag{3.2}$$

Here $\vec{F}(u)$ is a vector-valued function of a single variable.

**a)** Use the boundary condition at $\sigma = \sigma_1$ to derive a periodicity condition for $\vec{F}$.

**b)** Derive the constraints on $\vec{F}(u)$ that arise from the parametrisation conditions

$$\left(\frac{\partial \vec{X}}{\partial \sigma} \pm \frac{1}{c}\frac{\partial \vec{X}}{\partial t}\right)^2 = 1 \,. \tag{3.3}$$

Consider now a kid using a relativistic open string as a jumping rope. He holds the endpoints in his hands, fixed at $\vec{x}_1 = (0,0,0)$ and $\vec{x}_2 = (0,0,L_0)$ (in Cartesian coordinates). As he starts jumping, we observe that at the origin the vector $\vec{X}'$ tangent to the string rotates with constant angular velocity around the $z$ axis, forming an angle $\gamma$ with it. The kid is depicted in fig. 1.

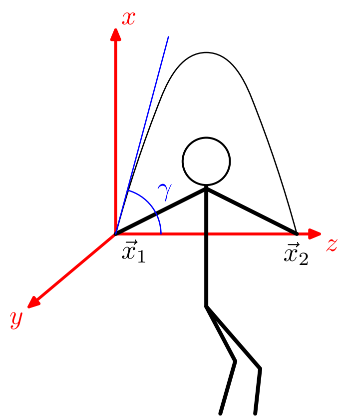


Figure 1: Kid jumping a relativistic rope

**c)** With the above data, derive the explicit expression of $\vec{F}'(u)$.

**d)** Exploit the periodicity of $\vec{F}$ and the explicit form of $\vec{F}'$ to find $\sigma_1$ in terms of $\gamma$ and $L_0$.

**e)** Calculate explicitly $\vec{X}(t,\sigma)$.

**f)** Using the above explicit form of $\vec{X}(t,\sigma)$, explain how the energy is distributed in the string along the $z$ axis.

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## 3.2. Classical closed strings and cusps

Let us consider closed strings in $D$-dimensional Minkowski space. We fix static gauge $X^0 = R\tau$; in this gauge it is possible to choose a parametrization of the worldsheet such that $\vec{X}$ satisfies the wave equation. The classical solution for the wave equation can be then expressed as

$$\vec{X}(\sigma,\tau) = \frac{1}{2}\left[\vec{X}_L(\tau+\sigma) + \vec{X}_R(\tau-\sigma)\right]. \tag{3.4}$$

The solution must satisfy the constraints

$$\dot{\vec{X}} \cdot \vec{X}' = 0\,, \qquad \dot{\vec{X}}^2 + \vec{X}'^2 = R^2\,. \tag{3.5}$$

**a)** Show that the solution

$$\begin{aligned} X^1 &= R\cos(\sigma)\cos(\tau)\,, \\ X^2 &= R\cos(\sigma)\sin(\tau)\,, \\ X^i &= 0\,, \quad i \geq 3\,, \end{aligned} \tag{3.6}$$

can be rewritten in the form of eq. (3.4) *and* that it satisfies the constraints of eq. (3.5). Moreover, calculate the energy $P^0$ and the angular momentum $J_{ij}$ of the solution.

**b)** Show that the solution

$$\begin{aligned} X^1 &= R\cos(\sigma)\cos(\tau)\,, \\ X^2 &= R\cos(2\sigma)\sin(2\tau)\,, \\ X^i &= 0\,, \quad i \geq 3\,, \end{aligned} \tag{3.7}$$

can be rewritten in the form of eq. (3.4) but that it does *not* satisfies the constraints of eq. (3.5).

**c)** Closed strings can develop *cusps*. A cusp is a configuration where the parametrisation is singular at some point, that is

$$\frac{\partial \vec{X}}{\partial \sigma}(\sigma_0, t) = 0\,. \tag{3.8}$$

With slight abuse of notation, we call the point $\sigma_0$ itself the cusp.
Recall that for a closed string we have the periodicity condition $\vec{X}(\sigma+\sigma_1) = \vec{X}(\sigma)$ for some $\sigma_1$. Show that the string reaches the speed of light at a cusp; moreover, show that cusps will appear and disappear periodically along the string.

**d)** The discussion up to now has been quite general and has not referred to any specific spacetime dimension. The formation of cusps however takes place generically in $3+1$ dimensions, but not in higher dimensions. Explain why. *Hint:* what are the trajectories of $\frac{\partial \vec{X}_{L/R}}{\partial \sigma^{\pm}}$, where $\sigma^{\pm} = \sigma \pm \tau$?