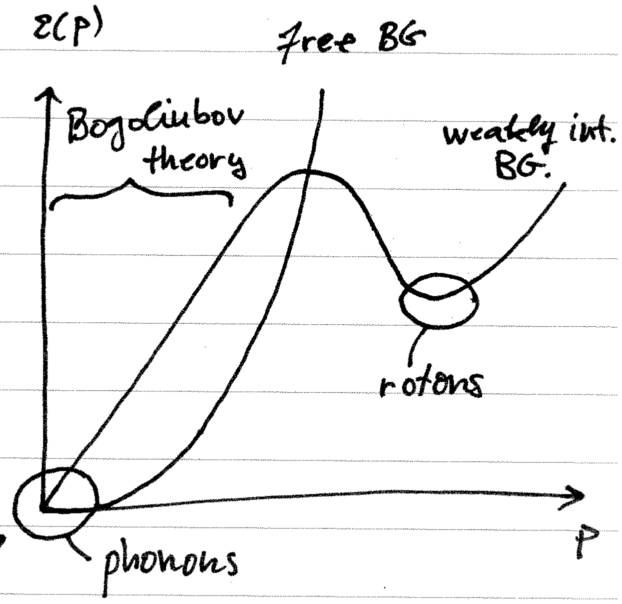
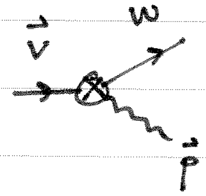
Bogoliubov theory of a weakly interacting Bose gas

Under normal pressure, the liquid ^4He undergoes a transition to a new superfluid phase as $T \to 0$. This phase is characterized by many striking properties, most notably that it is completely frictionless. As was shown by Bogoliubov, this can only be explained in the presence of (weak) interaction between the Bose particles forming the gas. For example, the BEC in the case of a free Bose gas does not show a superfluid phase.

The condition for the existence of such a phase can be obtained by looking at the energy spectrum of the free and weakly int. Bose gas. The crucial difference is the slope of $\varepsilon(p)$ as $p \to 0$, which indicates fundamental differences in the excitation spectrum.



When does a fluid flows frictionlessly? Consider a defect (= a wall) with mass M moving at speed $\vec{v}$ w.r.t. the fluid. Friction comes into the game when the defect is able to ~~scatter with particles in the fluid~~ create excitations in the fluid, transfering energy, via an <u>inelastic scattering</u>:



$$\underbrace{M\vec{v}}_{\text{defect}} \longrightarrow \underbrace{M\vec{w}}_{\text{defect}} + \underbrace{\vec{p}}_{\text{excitation}} \qquad (*)$$

Such an inelastic scattering is favored if it reduces the energy; i.e. if we have

(Some energy is absorbed by the defect → inelastic)
$$\varepsilon(\vec{p}) + \frac{M}{2}\vec{w}^2 \leq \frac{M}{2}\vec{v}^2$$

Plugging momentum conservation (*) in the above equation we get

$$\varepsilon(\vec{p}) + \frac{M}{2}\vec{v}^2 - \vec{p}\cdot\vec{v} + \frac{\vec{p}^2}{2M} \leq \frac{M}{2}\vec{v}^2$$

and neglecting the term $\vec{p}^2/2M$ (low momentum exchange compared to defect mass) we obtain the Landau condition:

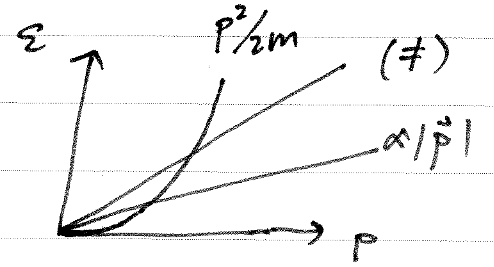
$$\boxed{\varepsilon(\vec{p}) - \vec{p}\cdot\vec{v} \leq 0} \qquad (‡)$$

If this condition can be fullfilled, then the fluid can exchange energy with the defect and there is friction. On the other hand if the condition is not fulfilled, then the fluid is frictionless.

Choosing $\vec{p} \parallel \vec{v}$ we can now see why different behaviors of $\varepsilon(\vec{p})$ at $\vec{p}=0$, influence superfluidity:

*) Free Bose gas: $\varepsilon(\vec{p}) = \frac{\vec{p}^2}{2m}$ ⇒ (‡) never fulfilled
⇒ not superfluid

*) Weakly int. Bose gas: $\varepsilon(\vec{p}) \propto |\vec{p}|$ ⇒ (‡) fulfilled
⇒ superfluid



We now turn to the Bogoliubov theory that is able to predict the linear dependence of $\varepsilon(\vec{p})$ for low $\vec{p}$'s. We consider the following second quantized Hamiltonian with contact interactions:

$$\mathcal{H} = \sum_{k} (\varepsilon(k) - \mu) \hat{a}_k^+ \hat{a}_k + \frac{U}{2V} \sum_{k,k',q} \hat{a}_{k+q}^+ \hat{a}_{k'-q}^+ \hat{a}_{k'} \hat{a}_k$$

We now turn to the Bogoliubov approximation (see tutorial 8) by identifying $\hat{a}_0^+$ and $\hat{a}_0$ with $\sqrt{N_0}$, where N_0 denotes the total number of particles in the condensate. This is justified if $N_0 \gg N - N_0$. Expanding $\mathcal{H}$ in this limit we obtain:

$$\mathcal{H}' = \sum_{k \neq 0} \left(\varepsilon_k - \mu + \frac{UN_0}{V} \right) \hat{a}_k^+ a_k \overbrace{- N_0 \mu + \frac{UN_0^2}{2V}}^{\Rightarrow\ \mu = \frac{UN_0}{V}}$$

$$+ \sum_{k \neq 0} \frac{UN_0}{2V} \left\{ \hat{a}_k^+ \hat{a}_{-k}^+ + \hat{a}_{-k} \hat{a}_k + \hat{a}_k^+ a_k + \hat{a}_{-k}^+ \hat{a}_{-k} \right\}$$

$$+ \mathcal{O}(N_0^{1/2})$$

And setting $\mu = \frac{UN_0}{V}$ (from $\left.\frac{\partial \mathcal{H}}{\partial N_0}\right|_{T=0} = -\mu + \frac{UN_0}{V} = 0$) we get ($n_0 = N_0/V$)

$$\mathcal{H}' = \frac{1}{2} \sum_{k \neq 0} \Big\{ \overbrace{(\varepsilon_k + Un_0)}^{\equiv \tilde{\varepsilon}_k} (\hat{a}_k^+ \hat{a}_k + a_{-k}^+ a_{-k})$$

$$+ Un_0 \underbrace{(\hat{a}_k^+ a_k^+ + \hat{a}_{-k} \hat{a}_k)}_{} \Big\}$$

$$- N_0^2 \frac{U}{2V}$$

absorption / creation of quasi-particle states (Bogoliubons!) into / from the condensate!

Because of the creation / anihilation operators we do not know how to compute the spectrum of this Hamiltonian. The trick is to write:

$$\hat{a}_k = u_k \hat{\gamma}_k - v_k \hat{\gamma}^+_{-k} \quad , \quad \hat{a}_{-k} = u_k \hat{\gamma}_{-k} - v_k \gamma^+_k$$

where $\hat{\gamma}_k$ and $\hat{\gamma}^+_k$ are new operators and $u_k, v_k \in \mathbb{C}$. Since the quasi-particles must be of bosonic nature (b/c the gas is made of "fundam." Bose particles) we require

$$[\hat{\gamma}_k, \hat{\gamma}^+_{k'}] = \delta_{kk'} \, , \quad [\hat{\gamma}_k, \hat{\gamma}_{k'}] = [\hat{\gamma}^+_k, \hat{\gamma}^+_{k'}] = 0 \, .$$

Plugging this into the ~~definition of~~ commutation relations for the a_k we get

$$[\hat{a}_k, \hat{a}^+_k] = [u_k \hat{\gamma}_k - v_k \hat{\gamma}_k, \bar{u}_k \hat{\gamma}^+_{-k} - v_k \hat{\gamma}_{-k}]$$

$$= |u_k|^2 \underbrace{[\hat{\gamma}_k, \hat{\gamma}^+_k]}_{=1} + |v_k|^2 \underbrace{[\hat{\gamma}^+_{-k}, \hat{\gamma}_{-k}]}_{=1}$$

$$- u_k \bar{v}_k \underbrace{[\hat{\gamma}_k, \hat{\gamma}_{-k}]}_{=0} - u^*_k v_k \underbrace{[\hat{\gamma}^+_{-k}, \hat{\gamma}^+_k]}_{=0}$$

$$= |u_k|^2 - |v_k|^2 \overset{!}{=} \underline{1} \, . \qquad (*)$$

All the other relations for $\hat{a}_k$ and a^+_k are automatically fulfilled. Hence the operators $\hat{\gamma}_k$ and $\hat{\gamma}^+_k$ can be used instead of $\hat{a}_k$ and $\hat{a}^+_k$ if $(*)$ is satisfied.

Now we show how these operators diagonalize the Hamiltonian $\mathcal{H}'$. Plugging the definitions of $\hat{a}_k$ and $\hat{a}_k$ we get

$$\mathcal{H}' = \frac{1}{2}\sum_{k\neq 0} h_k - \frac{U N_0^2}{2V}$$

with
$$\begin{aligned} h_k = {} & \tilde{\varepsilon}_k \Big[(\bar{u}_k \hat{\gamma}_k^+ - \bar{v}_k \hat{\gamma}_{-k})(u_k \hat{\gamma}_k - v_k \hat{\gamma}_{-k}^+) \\ & + (\bar{u}_k \hat{\gamma}_{-k}^+ - \bar{v}_k \hat{\gamma}_k)(u_k \hat{\gamma}_{-k} - v_k \hat{\gamma}_k^+) \Big] \\ & + U n_0 \Big[(\bar{u}_k \hat{\gamma}_k^+ - \bar{v}_k \hat{\gamma}_{-k})(\bar{u}_k \hat{\gamma}_{-k}^+ - \bar{v}_k \hat{\gamma}_k) \\ & \qquad + (u_k \hat{\gamma}_k - v_k \hat{\gamma}_{-k}^+)(u_k \hat{\gamma}_{-k} - v_k \hat{\gamma}_k^+) \Big] \end{aligned}$$

$[\hat{\gamma}_k, \hat{\gamma}_k^+] = 1$

$$= \Big[\overbrace{\tilde{\varepsilon}_k (|u_k|^2 + |v_k|^2) - U n_0 (\bar{u}_k \bar{v}_k + u_k v_k)}^{\equiv E_k} \Big]$$

diagonal (good) $\longrightarrow \times (\hat{\gamma}_k^+ \hat{\gamma}_k + \hat{\gamma}_{-k}^+ \hat{\gamma}_{-k})$

non diagonal (bad)
$$\left\{ \begin{aligned} & + (\hat{\gamma}_k^+ \hat{\gamma}_{-k}^+) \left[-2 \bar{u}_k v_k \varepsilon_k + U n_0 (\bar{u}_k^2 + v_k^2) \right] \\ & + (\hat{\gamma}_k \hat{\gamma}_{-k}) \left[-2 u_k \bar{v}_k \varepsilon_k + U n_0 (u_k^2 + \bar{v}_k^2) \right] \end{aligned} \right.$$

In order for the non-diagonal terms to vanish we require that both coefficients vanish. If we set u_k and v_k to be real valued, it means

$$-2 u_k v_k \tilde{\varepsilon}_k + U n_0 (u_k^2 + v_k^2) \overset{!}{=} 0 \qquad (*)$$

under the condition that $v_k^2 - u_k^2 = 1$.

This later condition is solved for real valued u_k and v_k if we set

$$u_k = \frac{1}{\sqrt{1-x_k^2}} , \quad v_k = \frac{x_k}{\sqrt{1-x_k}}$$

for $x^2 < 1$. Plugging this into eq. (*) we get:

$$0 \overset{!}{=} -\frac{2x_k}{1-x_k^2}\tilde{\varepsilon}_k + \frac{1+x_k^2}{1-x_k^2} U n_0 \qquad (\text{ζ})$$

Recall:
$\tilde{\varepsilon}_k = \varepsilon_k + U n_0$

whose solution is given by:

$$x_k = \frac{\tilde{\varepsilon}_k \pm \sqrt{\tilde{\varepsilon}_k^2 - (U n_0)^2}}{U n_0} = \frac{\tilde{\varepsilon}_k}{U n_0} \pm \sqrt{\left(\frac{\tilde{\varepsilon}_k}{U n_0}\right)^2 - 1}$$

Note that we can choose the "minus" solution since ~~either~~ the "plus" solution would just correspond to $u_k \leftrightarrow v_k$.

Using this we can now compute E_k explicitly and hence obtain the dispersion relation. We set $e \equiv \tilde{\varepsilon}_k / U n_0$ such that

$$\frac{E_k}{U n_0} = e\,(u_k^2 + v_k^2) - (u_k v_k + u_k v_k)$$

$$= e\left(\frac{1}{1-x_k^2} + \frac{x_k^2}{1-x_k^2}\right) - \frac{2x_k}{1-x_k^2}$$

Note that eq. (ζ) implies $\frac{2x_k}{1-x_k^2} e = \frac{1+x_k^2}{1-x_k}$ such that

$$= (e^2 - 1)\frac{x_k}{1-x_k^2}$$

$$\chi_k = e-\sqrt{e^2-1} \overset{\vee}{=} (e^2-1)\,\frac{2(e-\sqrt{e^2-1})}{1-e^2+2e\sqrt{e^2-1}-e^2+1} \Big\} = 2(1-e^2+e\sqrt{e^2-1}$$

$$= \frac{e^2-1}{\sqrt{e^2-1}} \cdot \frac{e-\sqrt{e^2-1}}{e-\sqrt{e^2-1}} = \sqrt{e^2-1}$$

such that we get :

$$E_k = Un_0\sqrt{e^2-1} = \sqrt{\tilde{\varepsilon}_k - U^2 n_0^2}$$

$$= \sqrt{(\varepsilon_k + Un_0)^2 - U^2 n_0^2}$$

$$= \sqrt{\varepsilon_k^2 + 2Un_0\varepsilon_k} \cong \sqrt{2Un_0\varepsilon_k} \propto |\vec{p}|$$

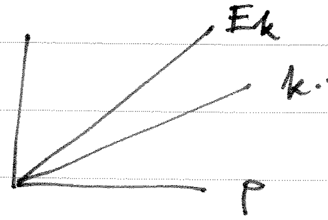
as $\vec{k} \to 0$.

Hence using $\varepsilon_k = \frac{k^2}{2m}$ we get $E_k \sim \sqrt{\frac{Un_0}{m}}\,|\vec{k}|$ which is a linear dispersion relation! Going back to the Landau criterion we get

$$E_k - |\vec{k}|V \le 0 \overset{\vec{k}\to 0}{\Longrightarrow} \sqrt{\frac{Un_0}{m}}\,|\vec{k}| - |\vec{k}|V \le 0$$

Such that the fluid is frictionless for velocities smaller than

$$V < V_c = \sqrt{\frac{Un_0}{m}}$$



$\Rightarrow$ superfluidity!