Second quantization

Occupation number representation for independent indistinguishable particles

$$\mathcal{H}|\psi_\nu\rangle = \epsilon_\nu|\psi_\nu\rangle$$ stationary states

Hilbert space of *N* particles $\mathcal{Q}_N$

Fock space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{Q}_n$$

$$|n_{\nu_1}, n_{\nu_2}, \ldots, n_\nu, \ldots\rangle \quad \begin{cases} E = \sum_\nu \epsilon_\nu n_\nu \\ N = \sum_\nu n_\nu \end{cases}$$

n_ν: number of partcles in state $|\psi_\nu\rangle$

creation / annihilation operators

$$\hat{a}_\nu^\dagger : \mathcal{Q}_N \to \mathcal{Q}_{N+1} \qquad \hat{a}_\nu^\dagger|n_{\nu_1}, \ldots, n_\nu, \ldots\rangle = \sqrt{n_\nu + 1}|n_{\nu_1}, \ldots, n_\nu + 1, \ldots\rangle$$

$$\hat{a}_\nu : \mathcal{Q}_N \to \mathcal{Q}_{N-1} \qquad \hat{a}_\nu|n_{\nu_1}, \ldots, n_\nu, \ldots\rangle = \sqrt{n_\nu}|n_{\nu_1}, \ldots, n_\nu - 1, \ldots\rangle$$

Second quantization

creation / annihilation operators

$$\hat{a}^\dagger_\nu : \mathcal{Q}_N \to \mathcal{Q}_{N+1} \qquad \hat{a}^\dagger_\nu |n_{\nu_1}, \dots, n_\nu, \dots\rangle = \sqrt{n_\nu + 1}|n_{\nu_1}, \dots, n_\nu + 1, \dots\rangle$$

$$\hat{a}_\nu : \mathcal{Q}_N \to \mathcal{Q}_{N-1} \qquad \hat{a}_\nu |n_{\nu_1}, \dots, n_\nu, \dots\rangle = \sqrt{n_\nu}|n_{\nu_1}, \dots, n_\nu - 1, \dots\rangle$$

Bosons: $[\hat{a}_\nu, \hat{a}^\dagger_{\nu'}] = \delta_{\nu\nu'} \qquad [\hat{a}_\nu, \hat{a}_{\nu'}] = [\hat{a}^\dagger_\nu, \hat{a}^\dagger_{\nu'}] = 0$

$$|n_{\nu_1}, \dots, n_\nu, \dots\rangle = \frac{\cdots (\hat{a}^\dagger_\nu)^{n_\nu} \cdots (\hat{a}^\dagger_{\nu_1})^{n_{\nu_1}}}{\sqrt{n_{\nu_1}! \cdots}} |0\rangle$$

Fermions: $\{\hat{a}_\nu, \hat{a}^\dagger_{\nu'}\} = \delta_{\nu\nu'} \qquad \{\hat{a}_\nu, \hat{a}_{\nu'}\} = \{\hat{a}^\dagger_\nu, \hat{a}^\dagger_{\nu'}\} = 0$

$$|n_{\nu_1}, \dots, n_\nu, \dots\rangle = \cdots (\hat{a}^\dagger_\nu)^{n_\nu} \cdots (\hat{a}^\dagger_{\nu_1})^{n_{\nu_1}} \; |0\rangle$$

Second quantization

number operator of state $|\psi_\nu\rangle$

$$\hat{n}_\nu = \hat{a}^\dagger_\nu \hat{a}_\nu$$

$$\mathcal{H} = \sum_\nu \epsilon_\nu \hat{n}_\nu = \sum_\nu \epsilon_\nu \hat{a}^\dagger_\nu \hat{a}_\nu$$

$$\hat{N} = \sum_\nu \hat{n}_\nu = \sum_\nu \hat{a}^\dagger_\nu \hat{a}_\nu$$

field operators

$$\psi_\nu(\vec{r}) = \langle \vec{r} | \psi_\nu \rangle$$

single-particle
wavefunction

$$\hat{\Psi}(\vec{r}) = \sum_\nu \psi_\nu(\vec{r}) \hat{a}_\nu$$

$$\hat{\Psi}(\vec{r})^\dagger = \sum_\nu \psi^*_\nu(\vec{r}) \hat{a}^\dagger_\nu$$

$$[\hat{\Psi}(\vec{r}), \hat{\Psi}(\vec{r}\,')^\dagger] = \delta(\vec{r} - \vec{r}\,') \qquad [\hat{\Psi}(\vec{r}), \hat{\Psi}(\vec{r}\,')] = [\hat{\Psi}(\vec{r})^\dagger, \hat{\Psi}(\vec{r}\,')^\dagger] = 0 \qquad \text{B}$$

$$\{\hat{\Psi}(\vec{r}), \hat{\Psi}(\vec{r}\,')^\dagger\} = \delta(\vec{r} - \vec{r}\,') \qquad \{\hat{\Psi}(\vec{r}), \hat{\Psi}(\vec{r}\,')\} = \{\hat{\Psi}(\vec{r})^\dagger, \hat{\Psi}(\vec{r}\,')^\dagger\} = 0 \qquad \text{F}$$

Second quantization

real-space state

$$|\vec{r}_1, \ldots, \vec{r}_N\rangle = \frac{1}{\sqrt{N!}} \hat{\Psi}(\vec{r}_N)^\dagger \cdots \hat{\Psi}(\vec{r}_1)^\dagger |0\rangle$$

many-body wave function

$$\Phi(\vec{r}_1, \ldots, \vec{r}_N) = \langle \vec{r}_1, \ldots, \vec{r}_N | n_{\nu_1}, n_{\nu_2}, \ldots \rangle$$

operators

$$\hat{\rho}(\vec{r}) = \hat{\Psi}(\vec{r})^\dagger \hat{\Psi}(\vec{r})$$ particle density

$$\hat{\vec{J}}(\vec{r}) = \frac{\hbar}{2mi} \left\{ \hat{\Psi}(\vec{r})^\dagger \left(\vec{\nabla} \hat{\Psi}(\vec{r}) \right) - \left(\vec{\nabla} \hat{\Psi}(\vec{r})^\dagger \right) \hat{\Psi}(\vec{r}) \right\}$$ current density