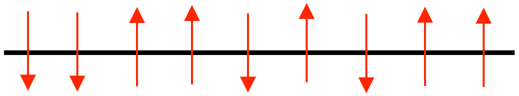
Ising model - mean field approximation

Ising model:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - \sum_i s_i H \qquad s_i = \pm s$$

nearest neighbor



rewrite: $s_i = \langle s_i \rangle + (s_i - \langle s_i \rangle) = m + (s_i - m) = m + \delta s_i$

$$\begin{aligned}
\Rightarrow \quad \mathcal{H} &= -J \sum_{\langle i,j \rangle} \{m + (s_i - m)\}\{m + (s_j - m)\} - \sum_i s_i H \\
&= -J \sum_{\langle i,j \rangle} \{m^2 + m(s_i - m) + m(s_j - m) + \delta s_i \delta s_j\} - \sum_i s_i H \\
&= -J \sum_i \left(z m s_i - \frac{z}{2} m^2 \right) - \sum_i s_i H - J \sum_{\langle i,j \rangle} \delta s_i \delta s_j
\end{aligned}$$

"small"

Ising model - mean field approximation

fluctuations small: $$\frac{\langle \delta s_i \delta s_j \rangle}{\langle s_i \rangle \langle s_j \rangle} = \frac{\langle \delta s_i \delta s_j \rangle}{m^2} \ll 1$$

↓

mean field approximation $$\mathcal{H}_{mf} = -\sum_i s_i h_{\text{eff}} + NJ\frac{z}{2}m^2$$

ideal paramagnet with effectiv field $$h_{\text{eff}} = Jzm + H$$

➡ canonical ensemble

$$Z(T, m, H) = e^{-\beta J z m^2 N/2} \left\{2\cosh(\beta s h_{\text{eff}})\right\}^N$$

free energy

$$F(T, H, m) = NJ\frac{z}{2}m^2 - Nk_B T \ln\left\{2\cosh(\beta s h_{\text{eff}})\right\}$$

$$F(T,H,m) = NJ\frac{z}{2}m^2 - Nk_BT\ln\{2\cosh(\beta s h_{\text{eff}})\}$$

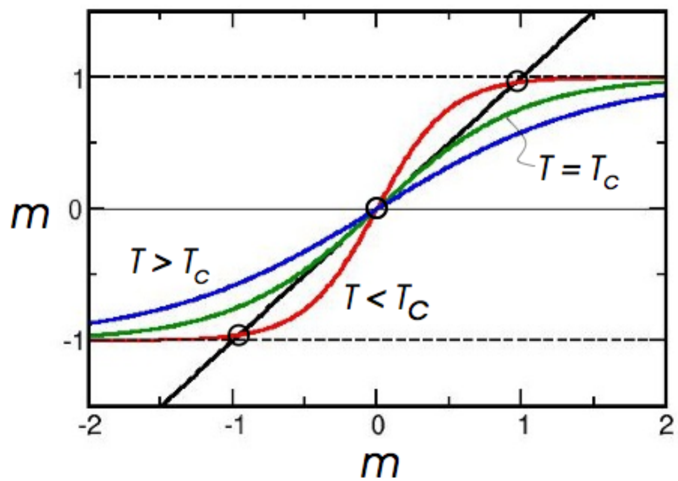
minimize free energy w.r.t. m

$$0 = \frac{\partial F}{\partial m} = NJzm - NJzs\tanh(\beta s h_{\text{eff}})$$

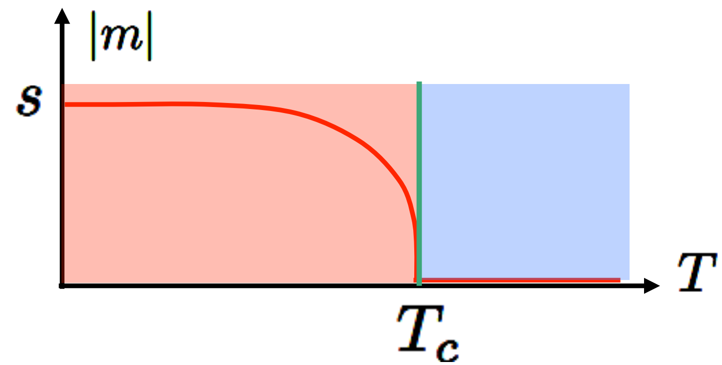
self-consistency →

$$m = \langle s_i \rangle = s\tanh(\beta s h_{\text{eff}})$$

$H = 0$



critical temperature $T_c = \frac{Jzs^2}{k_B}$



Ising model - mean field approximation

$$F(T,H,m) = NJ\frac{z}{2}m^2 - Nk_BT\ln\{2\cosh(\beta s h_{\text{eff}})\}$$

$T \to T_{c-}$ ➡ $m \to 0$ ➡ expand free energy

$$F(T,H=0,m) \approx F_0(T) + NJz\left[\left(\frac{T}{T_c}-1\right)\frac{m^2}{2} + \frac{m^4}{12s^2}\right]$$

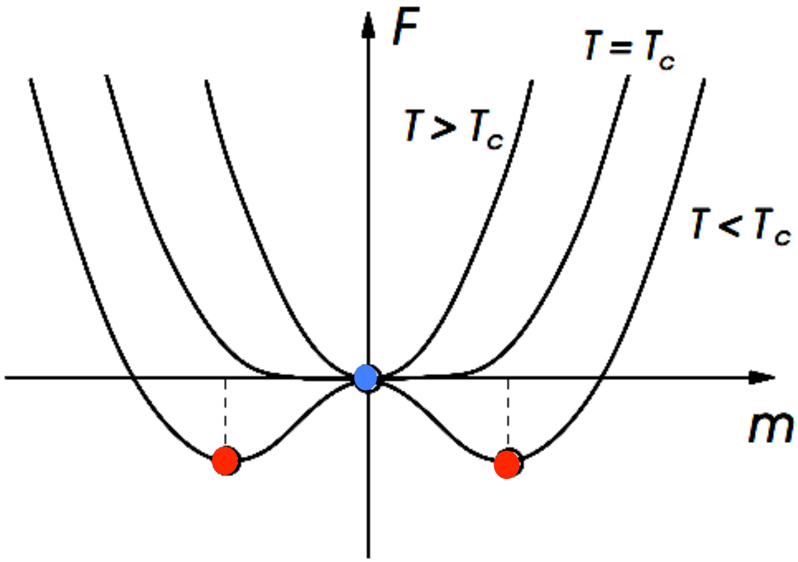
Landau theory

$$m(T) = \begin{cases} \pm s\sqrt{3\tau} & T < T_c \\ 0 & T \geq T_c \end{cases}$$

$$\tau = 1 - T/T_c$$

order parameter grows continuously

2nd-order phase transition



Ising model - mean field approximation

$$F(T) = F_0(T) - \frac{3Nk_BT_c\tau^2}{4}\Theta(\tau)$$

entropy

$$S(T) = -\frac{\partial F(T)}{\partial T}$$

$$= Nk_B \ln 2 - \frac{3Nk_B\tau}{2}\Theta(\tau)$$

heat capacity

$$\frac{C}{T} = \frac{\partial S}{\partial T} = \frac{3Nk_B}{2T_c}\Theta(\tau) + C_0$$

$$\tau = 1 - T/T_c$$