Momentum distribution for Fermions and Bosons 1

$$\mathcal{Z} = \prod_{\vec{p}'} \sum_{n_{\vec{p}'}} \left(ze^{-\beta\epsilon_{\vec{p}'}}\right)^{n_{\vec{p}'}} = \begin{cases} \prod_{\vec{p}'} \left(1 + ze^{-\beta\epsilon_{\vec{p}'}}\right) & \text{fermions} \\ \prod_{\vec{p}'} \dfrac{1}{1 - ze^{-\beta\epsilon_{\vec{p}'}}} & \text{bosons} \end{cases}$$

$$\langle n_{\vec{p}} \rangle = \frac{1}{\mathcal{Z}} \left\{ \prod_{\vec{p}' \neq \vec{p}} \sum_{n_{\vec{p}'}} \left(ze^{-\beta\epsilon_{\vec{p}'}}\right)^{n_{\vec{p}'}} \right\} \sum_{n_{\vec{p}}} n_{\vec{p}} \left(ze^{-\beta\epsilon_{\vec{p}}}\right)^{n_{\vec{p}}}$$

$$= \frac{\sum_{n_{\vec{p}}} n_{\vec{p}} \left(ze^{-\beta\epsilon_{\vec{p}}}\right)^{n_{\vec{p}}}}{\sum_{n_{\vec{p}}} \left(ze^{-\beta\epsilon_{\vec{p}}}\right)^{n_{\vec{p}}}} = -k_B T \frac{\partial}{\partial \epsilon_{\vec{p}}} \ln \mathcal{Z}$$

Momentum distribution for Fermions and Bosons

Fermions

$$\langle n_{\vec{p}} \rangle = \frac{\sum_{n_{\vec{p}}} n_{\vec{p}} (z e^{-\beta \epsilon_{\vec{p}}})^{n_{\vec{p}}}}{\sum_{n_{\vec{p}}} (z e^{-\beta \epsilon_{\vec{p}}})^{n_{\vec{p}}}} = \frac{z e^{-\beta \epsilon_{\vec{p}}}}{1 + z e^{-\beta \epsilon_{\vec{p}}}} = \boxed{\frac{1}{z^{-1} e^{\beta \epsilon_{\vec{p}}} + 1}}$$

$$n_{\vec{p}} = 0, 1$$

Fermi-Dirac distribution

Bosons

$$\langle n_{\vec{p}} \rangle = \frac{\sum_{n_{\vec{p}}} n_{\vec{p}} (z e^{-\beta \epsilon_{\vec{p}}})^{n_{\vec{p}}}}{\sum_{n_{\vec{p}}} (z e^{-\beta \epsilon_{\vec{p}}})^{n_{\vec{p}}}} = \frac{z e^{-\beta \epsilon_{\vec{p}}} (1 - z e^{-\beta \epsilon_{\vec{p}}})}{(1 - z e^{-\beta \epsilon_{\vec{p}}})^2} = \boxed{\frac{1}{z^{-1} e^{\beta \epsilon_{\vec{p}}} - 1}}$$

$$n_{\vec{p}} = 0, 1, \ldots, \infty$$

Bose-Einstein distribution

Additional details to Sect. 2.6.3

$$\mathcal{H}_Z = -g\frac{\mu_B}{\hbar}\sum_i \hat{s}_i^z H \qquad \{|\uparrow\rangle, |\downarrow\rangle\} \qquad g = 2$$

$$\mathcal{Z} = \prod_{\vec{p}} \left\{ \sum_{n_{\vec{p}}} \left(z e^{-\beta\epsilon_{\vec{p}} + \beta\mu_B H} \right)^{n_{\vec{p}}} \right\} \left\{ \sum_{n_{\vec{p}}} \left(z e^{-\beta\epsilon_{\vec{p}} - \beta\mu_B H} \right)^{n_{\vec{p}}} \right\}$$

$$= \prod_{\vec{p}} \prod_{\sigma=+,-} \sum_{n_{\vec{p}}} \left(z_\sigma e^{-\beta\epsilon_{\vec{p}}} \right)^{n_{\vec{p}}} \qquad z_\sigma = z e^{\beta\sigma\mu_B H}$$

$$\Omega = -k_B T \ln \mathcal{Z} = -\frac{V k_B T}{\lambda^3} \sum_\sigma f_{5/2}(z_\sigma)$$

Zeeman term in grand canonical partition function

$$m = -\frac{1}{V}\frac{\partial \Omega}{\partial H} = \frac{k_B T}{\lambda^3}\sum_\sigma \frac{\partial}{\partial H} f_{5/2}(z_\sigma) = \frac{k_B T}{\lambda^3}\sum_\sigma \frac{\partial z_\sigma}{\partial H}\ \frac{\partial}{\partial z} f_{5/2}(z)\bigg|_{z=z_\sigma}$$

$$= \frac{\mu_B}{\lambda^3}\sum_\sigma \sigma z_\sigma\ \frac{\partial}{\partial z} f_{5/2}(z)\bigg|_{z=z_\sigma} = \frac{\mu_B}{\lambda^3}\sum_\sigma \sigma f_{3/2}(z_\sigma)$$

using $\dfrac{\partial z_\sigma}{\partial H} = \sigma \mu_B \beta z_\sigma$

$$\chi = \frac{\partial m}{\partial H}\bigg|_{H=0} = \frac{\mu_B}{\lambda^3}\sum_\sigma \sigma\ \frac{\partial}{\partial H} f_{3/2}(z_\sigma)\bigg|_{H=0}$$

$$= \frac{\mu_B}{\lambda^3}\sum_\sigma \sigma \frac{\partial}{\partial z} f_{3/2}(z)\ \frac{\partial z_\sigma}{\partial H}\bigg|_{H=0} = \frac{2\mu_B^2}{\lambda^3 k_B T} f_{1/2}(z)$$