Exercise 1. ***Ising Model: Infinite-Range Forces and Mean Field.***

Consider an Ising model where now *all* spins interact between each other with the same strength $J = 1/N$ (long-range forces). The Hamiltonian is given by

$$\mathcal{H} = -\frac{1}{2N}\sum_{i,k} s_i s_k - H\sum s_i \ . \tag{1}$$

The coupling constant is rescaled by N so that the total energy remains finite; also the factor one-half compensates the fact that in the sum, each index i and k ranges independently from 1 to N, and thus we counted each bond twice.

In this exercise, we will show that the mean-field approach for this model is exact (at least for $N \to \infty$).

(a) In order to calculate the partition function for this model, we will introduce a little mathematical trick. Show that the Boltzmann factor which appears in the partition function can be written as

$$\mathrm{e}^{-\beta\mathcal{H}} = \sqrt{\frac{N\beta}{2\pi}}\int_{-\infty}^{\infty} d\lambda \exp\left(-\frac{N\beta\lambda^2}{2} + \sum_i \beta\left(\lambda + H\right)s_i\right) \ . \tag{2}$$

This is a particular case of the Gaussian transform method which will be seen in the lecture.

Hint. Remember the Gaussian integral $\int dx\, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}$ *and complete the square.*

(b) Show that the partition function can be written as

$$Z = \sqrt{\frac{N\beta}{2\pi}}\int d\lambda\, \mathrm{e}^{-N\beta A(\lambda)} \ , \qquad A\left(\lambda\right) = \frac{\lambda^2}{2} - \frac{1}{\beta}\ln\left(2\cosh\left[\beta(\lambda + H)\right]\right) \tag{3}$$

In order to determine the partition function, we will use the steepest descent method (a.k.a. Laplace method or saddle point approximation): the integral of the exponential is dominated by the maximum of the function in the exponential. Technically this is done by expanding the function in the exponent to second order at its maximum, and neglecting further orders.

(c) Determine the equation that λ should satisfy in order for it to be the maximum of the argument of the exponential.

Show that the partition function can be written (for large N) as

$$Z \approx \mathrm{e}^{-N\beta f} \ ; \qquad f = A\left(\lambda_0\right) + \frac{1}{2N\beta}\ln A''\left(\lambda_0\right) \approx A\left(\lambda_0\right) \ , \tag{4}$$

where f is the free energy per spin and λ_0 is the minimizer of the function $A\left(\lambda\right)$.

(d) Show that λ_0 is precisely the average magnetization of a spin, $\lambda_0 = \langle s_i \rangle =: m$. Deduce that your result coincides with the magnetization that you would get via mean field theory.

(e) When applying the Laplace method to calculate the partition function λ_0 must be the global minimizer of the function $A(\lambda)$. Convince yourself that for $H \neq 0$ the function $A(\lambda)$ has a unique global minimum. Do so by plotting $A(\lambda)$ for a fixed $H \neq 0$ and for different β. What is the error in the approximation if there are two identical minima?

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