Exercise 1. *Planets as black bodies?*

The Stefan-Boltzmann law states that the emission power per unit surface area of a black body reads

$$P_{em} = \sigma T^4 \quad \text{with} \quad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} \approx 5.6704 \cdot 10^{-8}\ \mathrm{Js^{-1}m^{-2}K^{-4}}. \tag{1}$$

(a) Making use of the Stefan-Boltzmann law, estimate the temperature of the Earth, Mars and Venus as if they were black bodies.

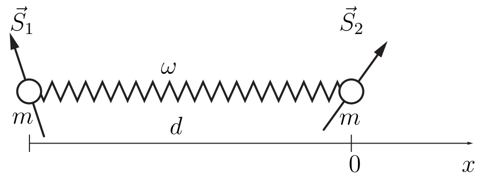
Hint. Compute the emission power of the sun as if it were a black body and assume that the energy emitted and absorbed by each planet has to balance.

(b) The correct results for the average temperatures are 288 K for the Earth, 218 K for Mars and 735 K for Venus. How do they compare with your estimates? What could be the reasons of the discrepancies?

Exercise 2. *Magnetostriction in a Spin-Dimer-Model.*

We consider a dimer consisting of two spin-1/2 particles with the Hamiltonian

$$\mathcal{H}_0 = J\left(\vec{S}_1 \cdot \vec{S}_2 + 3/4\right) ,$$



with $J > 0$. We already considered a dimer in exercise 4.2, but note that the energy levels are now shifted by a constant. This time, however, the distance between the two spins is not fixed and they are connected to each other by a spring. The spin–spin coupling constant depends on the distance between the two sites such that the Hamilton operator of the system is

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 + J(1-\lambda\hat{x})\left(\vec{S}_1 \cdot \vec{S}_2 + 3/4\right) , \tag{2}$$

where $\lambda \geq 0$, m is the mass of the two constituents, $m\omega^2$ is the spring constant, and x denotes the displacement from the equilibrium distance d between the two spins (defined for no spin-spin interaction).

(a) Calculate the canonical partition function, the internal energy, the specific heat and the entropy. Discuss the behavior of the entropy in the limit $T \to 0$ for different values of λ.

Hints. Rewrite the Hamiltonian using the total spin operator as in Exercise 4.2, and bring it by completing the square to the following form

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{X}^2 + \tilde{J}\,\hat{n}_t , \tag{3}$$

where $\hat{n}_t \equiv \vec{S}^2/2$ is the projector on the triplet subspace, $\hat{X}$ and $\tilde{J}$ are appropriately shifted quantities of $\hat{x}$ and J ($\hat{X}$ may depend on $\hat{n}_t$), and we have set $\hbar = 1$. Then note that $\hat{X}$ and $\hat{p}$ satisfy the same commutation relations as $\hat{x}$ and $\hat{p}$ such that the two first terms of (3) describe a quantum harmonic oscillator.

(b) Calculate the expectation value of the distance between the two spins, $\langle d + \hat{x} \rangle$, as well as $\langle (d + \hat{x})^2 \rangle$. How are these quantities affected by a magnetic field in z-direction, i.e., by adding an additional term in (2) of the form

$$\mathcal{H}_m = -g\mu_B H \sum_i \hat{S}_i^z \quad ?$$

Hints. Write first these expectation values in terms of $\langle \hat{n}_t \rangle$, which you can calculate explicitly. Recall that for a harmonic oscillator, $\langle \hat{X} \rangle$ vanishes, as well as $\langle \hat{a} \rangle$, $\langle \hat{a}^2 \rangle$ etc.

Then recalculate the partition function, adding the magnetic field term and see how this affects $\langle \hat{n}_t \rangle$.

(c) If the two sites are oppositely charged with charge $\pm q$, the dimer forms a dipole with moment $P = q \langle d + \hat{x} \rangle$. This dipole moment can be measured by applying an electric field E along the x-direction, resulting in the additional Hamiltonian term

$$\mathcal{H}_{\text{el}} = -q(d + \hat{x})E \ . \tag{4}$$

Calculate the susceptibility of the dimer at zero electric field,

$$\chi_0^{(\text{el})} = -\left.\frac{\partial^2 F}{\partial E^2}\right|_{E=0} , \tag{5}$$

and show that the simple form of the fluctuation-dissipation theorem, which asserts that

$$\chi_0^{(\text{el})} \propto \left\langle (d + \hat{x})^2 \right\rangle - \left\langle d + \hat{x} \right\rangle^2 , \tag{6}$$

is not valid here.

Hint. Redo the calculations from (a) with $\mathcal{H}_{el}$ included (but without magnetic field), by completing the square with a different definition of $\hat{X}$ and $\tilde{J}$.

(d) Proceeding as in Section 2.5.3 of the lecture notes, derives the correct fluctuation-dissipation theorem for this system.

Hint. Choosing the variable $\hat{X}$ from (c) as your fundamental degree of freedom introduces a dependence on E in $\hat{x}$, and hence in the coupling (4).

Office Hours: Monday, November 10, 14–16 PM (Romain Mueller, HIT K 21.3).