Exercise 3.1 Non-interacting Particles in the Gravitational Field

Consider a gas of non-interacting particles in the gravitational field

$$V_{\text{grav}}(x, y, z \geq 0) = mgz, \tag{1}$$

with the gravitational constant $g > 0$ at fixed temperature T. The volume of the gas is confined to a vertical, cylindrical vessel (radius R) of semi-infinite height.

a) Using the canonical ensemble, find the Helmholtz free energy, the entropy, and the internal energy of this system.

b) Consider the system from the viewpoint of a local thermal equilibrium. Find the local particle density at height z, $n(z)$, normalized such that $N = \int \mathrm{d}^3 q n(q_z)$. Find the local pressure $p(z)$ as well as the local internal energy density $u(z)$. Express $p(z)$ and $u(z)$ in terms of $n(z)$ to find the local caloric and thermal equations of state.

c) Calculate the heat capacity using

   i) the entropy found in a).
   ii) the equipartition law for $U = \langle \mathcal{H} \rangle$.
   iii) the local caloric and thermal equations of state.
   iv) the variance $(\Delta\mathcal{H})^2$.

   Discuss and compare the different results.

   Hints for:

   ii) Rewrite the Hamiltonian such that the equipartition law as given in the lecture notes may be applied directly.
   iii) Keeping the specific volume $v(z) = N/n(z)$ constant is equivalent to a constant local density and thus the we can write for the specific heat capacity

$$c_p = \frac{C_p}{N} \left(\frac{\partial u}{\partial T}\right)_n + \left\{ \left(\frac{\partial U}{\partial V}\right)_T + p \right\} \alpha = \left(\frac{\partial u}{\partial T}\right)_n + T \left(\frac{\partial p}{\partial T}\right)_n \alpha, \tag{2}$$

   where $\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p$ is the (local) thermal expansion coefficient. First, show that even with the local viewpoint α is given by $1/T$, independently of z.

   iv) Show that the kinetic and potential contributions to the variance separate as

$$(\Delta\mathcal{H})^2 = (\Delta\mathcal{H}_{\text{k}in})^2 + (\Delta\mathcal{H}_{\text{pot}})^2, \tag{3}$$

   and use the relation between the fluctuation of the energy and the heat capacity

$$(\Delta\mathcal{H})^2 = k_B T^2 C. \tag{4}$$

Exercise 3.2 Classical Ideal Paramagnet II

We consider an ideal paramagnet with magnetic moments pointing in arbitrary directions. The Hamiltonian has the usual Zeeman form

$$\mathcal{H} = -\sum_{i=1}^{N} \vec{m}^{(i)} \cdot \vec{H} = -\sum_{i=1}^{N} mH\cos\theta_i, \tag{5}$$

where the magnetic field $\vec{H} = (0, 0, H)$ points in the z-direction and the magnetic moments can be represented by two angles in spherical coordinates as

$$\vec{m}^{(i)}(\phi_i, \theta_i) = m(\cos\phi_i \sin\theta_i, \sin\phi_i \sin\theta_i, \cos\theta_i). \tag{6}$$

The phase-space of each moment consists of a unit sphere with volume element $d\Omega_i = \mathrm{d}\phi_i \mathrm{d}\theta_i \sin\theta_i = \mathrm{d}\phi_i \mathrm{d}\cos\theta_i$.

a) Using the canonical ensemble, compute the Helmholtz free energy, the internal energy, and the heat capacity. Does the heat capacity vanish as $T \to 0$? Interpret your result.

b) Compute the the magnetization $\langle \vec{m} \rangle$ and show that it satisfies the usual thermodynamical relation

$$\langle \vec{m} \rangle = -\frac{1}{N}\left(\frac{\partial F}{\partial \vec{H}}\right)_{T,N}. \tag{7}$$

c) Prove the fluctuation-dissipation theorem in this case, i.e.

$$(\Delta m_z)^2 = \frac{k_B T}{N}\chi_{zz}, \qquad \text{with} \quad \chi_{zz} = -\left(\frac{\partial^2 F}{\partial H_z^2}\right)_{T,N}, \tag{8}$$

where χ_{zz} is the magnetic susceptibility.

Office Hours: Monday, October 13, 2–4 PM (Romain Müller, HIT K 21.3).