Ginzburg criterion for mean field theories:

For a general GL functional, we consider fluctuations in the order parameter close to T_c. Here $|\psi| \ll 1$ and quartic terms can be neglected.

$$F = \int d^d x \left[a(T) |\psi|^2 + K |\nabla \psi|^2 \right]$$

Using $\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k \tilde{\psi}(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$

$$F = \sum_k \left[a(T) + K |\vec{k}|^2 \right] |\tilde{\psi}(\vec{k})|^2.$$

The Helmholtz free energy,

$$e^{-\frac{\mathcal{F}}{k_B T}} = \int D[\psi, \psi^*] \, e^{-\frac{F}{k_B T}}.$$

$D[\psi, \psi^*] \rightarrow$ functional integral measure.

Converting to the Fourier space variables,

$$D[\tilde{\psi}, \tilde{\psi}^*] = \prod_{\vec{q}} d\tilde{\psi}(\vec{q}) \, d\tilde{\psi}(-\vec{q}).$$

$$\equiv \prod_{\vec{q}} d\,\mathrm{Re}\,\tilde{\psi}(\vec{q}) \, d\,\mathrm{Im}\,\tilde{\psi}(\vec{q}).$$

For each $\vec{q}$ mode, the functional integral leads to a normal Gaussian integral.

The full functional integral is hence a product of all the integrals for the $\vec{q}$ modes.

$$e^{-\frac{\mathcal{F}(T)}{k_B T}} \equiv Z = \prod_{\vec{k}} \left[\frac{\pi k_B T}{a(T) + K|\vec{k}|^2} \right]$$

The resulting free energy arising from fluctuations is

$$f(T) = -k_B T \ln Z = -k_B T \sum_{\vec{k}} \ln \frac{\pi k_B T}{a + K|\vec{k}|^2}.$$

Specific heat $C_V = -T\mathcal{F}''(T)$

$$= k_B T^2 \sum_{\vec{k}} \frac{[a'(T)]^2}{[a(T) + K|\vec{k}|^2]^2} + \text{other terms} \quad (1)$$

Since we are interested in behaviour in the critical region, we only retain the most singular contributions in (1).

Now consider specific heat / unit volume $\Rightarrow$

$$C \equiv \frac{C_V}{k_B N} \qquad N \to \# \text{ of lattice sites}$$

$$N = \frac{V}{a^d} \qquad a \to \text{lattice spacing.}$$

$$C = T^2 a^d \int \frac{d^d k}{(2\pi)^d} \cdot \frac{[a'(T)]^2}{[a(T) + K|\vec{k}|^2]^2} + \cdots$$

* Note: While differentiating $f(T)$ for $C_V(T)$, you might encounter terms like $\sum_{\vec{k}}(\text{const!})$ in C_V. These are not really divergent as the integrals have a cut-off. Moreover, various other contributions at finite T have been neglected which would impact these "constant large terms".

Fluctuation contributions are divergent in other dimensions < 4 as $|t| \to 0$.

Mean field is ok if,

$$\frac{\alpha^2}{K^2} a^d \xi^{4-d} << 1 \qquad \text{Using } \xi = \left[\frac{K}{\alpha |t|}\right]^{1/2}.$$

we can define a temperature scale, t_G

such that $\frac{\alpha^2 a^d}{K^2} \left[\frac{K}{\alpha}\right]^{2-\frac{d}{2}} . |t_G|^{\frac{d}{2}-2} = 1$

$$\Rightarrow |t_G| = \left[\frac{a}{R}\right]^{\frac{2d}{4-d}} \quad \text{where } R = \sqrt{\frac{K}{\alpha}}.$$

A similar scale is obtained starting from the ordered side too.

Mean field ok if $|t| << \{t_G\}$.

For lattice ferromagnets in 3D $R \sim a \Rightarrow t_G \sim 1$, critical regime where mean field does not work is very large.

In conventional low-T SC. (s-wave).

$R \sim$ Cooper pair size $\sim (10^2 - 10^3)\, a$.

$\Rightarrow t_G \sim 10^{-18} \sim 10^{-12} \Rightarrow$ critical regime negligeably narrow and Mean field works!

$$a(T) = \frac{\alpha(T-T_c)}{T_c} . \qquad a'(T) = \frac{\alpha}{T_c}.$$

also, consider $T \gtrsim T_c$. $\quad a(T) > 0$.

$$C = \frac{\alpha^2 a^d}{K^2} \frac{T^2}{T_c^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{[\xi^{-2} + |\vec{k}|^2]^2}.$$

$$\xi = \sqrt{\frac{K}{|a|}} \propto |t|^{-1/2} \quad \text{where } t = \frac{T-T_c}{T_c}.$$

↓ coherence length in superconductors.

$$C(t) \approx \frac{\alpha^2 a^d}{K^2} \xi^4 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(1+\xi^2|\vec{k}|^2)^2}.$$

Changing variables, $\vec{q} = \xi \vec{k}$

$$C(t) \approx \frac{\alpha^2 a^d \xi^{4-d}}{K^2} \int_0^{\Lambda\xi} \frac{q^{d-1}\, dq\, d\Omega}{(2\pi)^d} \frac{1}{(1+q^2)^2}.$$

cut-off $\Lambda \sim \frac{1}{a}$ in real systems. [so no real divergence of integral]

__Leading "t" dependence__

$$C(t) \sim \begin{cases} \text{const} & d > 4 \\ -\ln|t| & d = 4 \\ |t|^{\frac{d}{2}-2} & d < 4. \end{cases}$$

Mean-field is qualitatively accurate in $d > 4$ with finite corrections

In unconventional superconductors, like high T_c, Cooper pairs are confined to smaller sizes and hence ξ decreases. For some materials like $YBa_2Cu_3O_{7-y}$ (the famous YBCO) with $T_c \sim 100\,K$ which implies

$\xi \sim 10\,\mathring{A} \Rightarrow t_G \sim 10^{-2}$ or 10^{-1}

there must be sizable deviations from such mean field predictions.