Introduction to Superconductivity
Exercise Sheet 9

HS14
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Long Josephson junctions

Having treated short Josephson junctions in the last Exercise Sheet 8, we now want to consider long Josephson junctions, where the spatial dimensions are bigger than the Josephson penetration depth*. In these junctions the self-fields cannot be neglected.*

Exercise 1: The sine-Gordon equation.

We are still using the setup described in Exercise Sheet 8, and we only consider situations where the field and the current do not depend on time.

You have derived the relation between the phase difference (mod 2π) and the field. The London penetration in y-direction remains just the same, so we can neglect it during the calculation. The field now also depends on z, as it includes the self-fields, and is no longer simply equal to the applied field B_0:

$$\frac{\partial \phi(z)}{\partial z} = \frac{2\pi h_{eff}}{\Phi_0} B_x(z). \tag{1}$$

When the effect of the induced self-fields cannot be neglected, we can combine the above relation with the current-phase relation and with Ampre's Law to find the so-called *sine-Gordon equation*

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\lambda_J^2} \sin \phi, \tag{2}$$

with the Josephson penetration depth given by

$$\lambda_J = \sqrt{\frac{c\Phi_0}{8\pi^2 J_c h_{eff}}}. \tag{3}$$

(a) Derive the sine-Gordon equation with the Josephson penetration depth yourself.

(b) Why is the Josephson penetration depth a *penetration depth*?

Comment: Linearize the sine-Gordon equation for small phases $\phi \ll 1$ and solve it.

(c) Why is the short Josephson junction ($w \ll \lambda_J$) equivalent to neglecting the self-fields?

(d) Which well-known differential equation has the same form as the sine-Gordon equation? Can you learn something about the solution of the sine-Gordon equation from the obvious solution of that equation?

(e) If a field B_0 is applied, what are the boundary conditions?

(f) What is the maximum field strength B_0 that can be applied?

(g) Analyze the sine-Gordon equation numerically.

Comment: This can be a one-liner for example in `Mathematica`.