BCS

$$H = \sum_{k\sigma} \epsilon_{k\sigma} c^{\dagger}_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{k,k'} b^{\dagger}_{k} b_{k'}$$

$$b^{\dagger}_{k} = c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow}$$

If $b^{\dagger} \to$ Cooper pair creation op.
then, in gs. for $V_{kk'} < 0 \to$ gs. will have no pair $(k\uparrow, -k\downarrow)$ occupied by a single $\bar{e}$.
Pair states are either empty or doubly occupied!
In this case, $\tilde{H} \to \sum_{k} 2\epsilon_k b^{\dagger}_{k} b_{k} + \sum_{kk'} V_{kk'} b^{\dagger}_{k} b_{k'}$

$\to$ looks like a bosonic hamiltonian!
However, b & $b^{\dagger}$ are not bosonic.

$$[b_k, b_{k'}] = [b^{\dagger}_{k}, b^{\dagger}_{k'}] = 0$$

$$[b_k, b^{\dagger}_{k'}] = [1 - \underbrace{c^{\dagger}_{k\uparrow} c_{k\uparrow} - c^{\dagger}_{-k\downarrow} c_{-k\downarrow}}_{\text{Pauli blocking!}}] \delta_{kk'}.$$

$+ (b^{\dagger}_{k})^2 = 0 \to$ so not bosonic.
(not easy to diagonalize etc.)

Mean field hypothesis!

$$b_k = \langle b_k \rangle + \underbrace{(b_k - \langle b_k \rangle)}_{\delta b_k}.$$

Neglecting terms proportional to $\delta b_k \delta b^{\dagger}_{k}$

$$H_{MF} = \sum_{k\sigma} \epsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{kk'} \left[\langle b_k \rangle b^{\dagger}_{k} + \langle b^{\dagger}_{k} \rangle b_k - \langle b^{\dagger}_{k} \rangle \langle b_k \rangle \right]$$

Rewrite as

$$H^{MF} = \sum_{k\sigma} \varepsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} + \sum_{k} \left[\Delta_k c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} + \Delta^{*}_{k} c_{-k\downarrow} c_{k\uparrow}\right)$$
$$- \sum_{kk'} V_{kk'} \langle b^{\dagger}_{k} \rangle \langle b_{k} \rangle$$

where
$$\boxed{\Delta_k = \sum_{k} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle}$$

- Notice that $[H^{MF}, N] \neq 0$ number not conserved!

$N = \sum_{k\sigma} c^{\dagger}_{k\sigma} c_{k\sigma}$ $\Rightarrow$ work in grand canonical ensemble $\boxed{K = H^{MF} - \mu N}$

Solution to MF equations.

$$K = \sum_{k} \left(c^{\dagger}_{k\uparrow} \; c_{-k\downarrow} \right) \begin{bmatrix} \xi_k & \Delta_k \\ \Delta^{*}_{k} & -\xi_k \end{bmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{\dagger}_{-k\downarrow} \end{pmatrix} + K_0$$

$\xi_k = \varepsilon_k - \mu$ $\qquad K_0 = \sum_{k} \xi_k - \sum_{kk'} V_{kk'} \langle c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} \rangle \langle h.c._{k'} \rangle$

↓ constant.

Diagonalize via unitary transform

$$\begin{pmatrix} c_{k\uparrow} \\ c^{\dagger}_{-k\downarrow} \end{pmatrix} = U_k \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{\dagger}_{-k\downarrow} \end{pmatrix} \qquad U_k = \begin{bmatrix} \cos\theta_k & -\sin\theta_k e^{i\varphi_k} \\ \sin\theta_k e^{-i\varphi_k} & \cos\theta_k \end{bmatrix}$$

with $\{ \gamma_{k\sigma}, \gamma^{\dagger}_{k'\sigma'} \} = \delta_{kk'} \delta_{\sigma\sigma'}$

This transformation mixes particle & hole states.

for each "k" mode

$\tilde{K} = U^{\dagger} K U$ & we want $\tilde{K}$ to be diagonal.

$\Rightarrow \phi = \arg(\Delta)$ and $\tilde{K} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$

$$2\xi \sin\theta \cos\theta = |\Delta|(\cos^2\theta - \sin^2\theta).$$

Bogoliubov-Valatin transf.

$$\tan 2\theta = \frac{|\Delta|}{\xi} \qquad \cos 2\theta = \frac{\xi}{E} \qquad \sin 2\theta = \frac{|\Delta|}{E}$$

$$E = \sqrt{\xi^2 + |\Delta|^2}$$

Restoring "k"

$$\phi_k = \arg(\Delta_k) \qquad \tan 2\theta_k = \frac{|\Delta_k|}{\xi_k}.$$

$$\cos 2\theta_k = \frac{\xi_k}{E_k} \qquad \sin 2\theta_k = \frac{|\Delta_k|}{E_k} \qquad E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$$

If Δ_k has only a weak dependence on k,

$E_k \to$ dispersion of excitations has minimum at $\xi_k = 0$ i.e. $k = k_F$

$|\Delta_k| \to$ superconducting gap!

$$\hat{K} = \sum_{k\sigma} E_k \gamma^{\dagger}_{k\sigma} \gamma_{k\sigma} + \sum_k (\xi_k - E_k) - \sum_{kk'} V_{kk'} \langle\rangle\langle\rangle$$

What is the ground state of such a system?

$\Rightarrow$ GS $\Rightarrow$ $\gamma_{k\sigma}|G\rangle = 0$

$$\Rightarrow \quad |G\rangle = \prod_k [\underbrace{\cos\theta_k}_{u_k} - \underbrace{\sin\theta_k e^{i\phi_k}}_{v_k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}] |0\rangle .$$

$\langle G|G\rangle = 1$.

Normal metallic GS $\rightarrow$ $|Q\rangle = \prod_{k\sigma}^{\epsilon_k<\mu} c^\dagger_{k\sigma}|0\rangle = \prod_k^{\epsilon_k<\mu} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}|0\rangle$

same form as $S\ \mathcal{C}\ |G\rangle$ with $u_k = 0$ $\epsilon_k<\mu$, $u_k = 1$ $\epsilon_k>\mu$, $|||^{ly}$ for v_k.

$$\gamma_{k\sigma} = \cos\theta_k c_{k\sigma} + \sigma \sin\theta_k e^{i\phi_k} c^\dagger_{-k-\sigma}$$

If $\Delta_k = 0$ & $\xi_k < 0$ for $k<k_F$, $\xi_k > 0$ for $k > k_F$

Choose $\phi_k = 0$.

$$\cos\theta_k = \theta(k-k_F) \qquad \sin\theta_k = \theta(k_F - k)$$

then $|G\rangle \longrightarrow$ describes filled Fermi sphere ~~with~~ upto $k = k_F$.

$k<k_f$ $\quad \gamma^\dagger_{k\sigma} = \sigma c_{-k-\sigma}$

$k>k_f$ $\quad \gamma^\dagger_{k\sigma} = c^\dagger_{k\sigma}$

elementary excns. are holes below k_F + electrons above k_F!

Self-consistency

$$N = \sum_{k\sigma} \langle c_{k\sigma}^{+} c_{k\sigma} \rangle$$

$$\Delta_k = \sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle .$$

We thus have

$$\langle c_{k\sigma}^{+} c_{k\sigma} \rangle = \cos^2\theta_k f_k + \sin^2\theta_k (1-f_f)$$

$$= \frac{1}{2} - \frac{\xi_k}{2E_k} \tanh\left(\beta \frac{E_k}{2}\right)$$

$$f_k = \langle \gamma_{k\sigma}^{\dagger} \gamma_{k\sigma} \rangle = \frac{1}{e^{\beta E_k} + 1} \longrightarrow \text{fermi fn. at temp } T = \frac{1}{\beta}.$$

$$\langle c_{-k-\sigma} c_{k\sigma} \rangle = \sigma \sin\theta_k \cos\theta_k e^{i\phi_k} [2f_k - 1]$$

$\sigma = \pm 1$

$$= \frac{-\sigma \Delta_k}{2E_k} \tanh\left(\beta \frac{E_k}{2}\right)$$

At T = 0

$$N = \sum_k \left[1 - \frac{\xi_k}{E_k} \right]$$

[th $\longrightarrow$ 1]

$$\Delta_k = -\sum_{k'} V_{kk'} \frac{\Delta_{k'}}{2E_{k'}}$$

$\longrightarrow$ BCS Gap Eq.

X^ model:

$$V_{kk'} = \begin{cases} -v/V & |\xi_k| < \hbar\omega_D, \ |\xi_{k'}| < \hbar\omega_D \\ 0 & \end{cases}$$

[like in Cooper pair problem]

$v > 0$ ⇒ X^ attractive in an energy band $\hbar\omega_D$.

$\omega_D \to$ Debye freq. for phonons

$$\Delta_k = \begin{cases} \Delta e^{i\phi} & |\xi_k| < \hbar\omega_D . \\ 0 & \text{otherwise} . \end{cases}$$

$\Delta \to$ real then.

$$\Delta = v \int \frac{d^3k}{(2\pi)^3} \frac{\Delta}{2E_k} \, \theta(\hbar\omega_D - |\xi_k|) .$$

assume $g(\varepsilon)$ varies slowly around $\mu \leq \varepsilon_F$. ($\mu \simeq \varepsilon_F$ true at low T).

$$= \frac{1}{2} v \, g(\varepsilon_F) \int_0^{\hbar\omega_D} d\xi \, \frac{\Delta}{\sqrt{\xi^2 + \Delta^2}}$$

Trivial soln: $\Delta = 0$!

Non-trivial soln:

$$1 = \frac{v g(\varepsilon_F)}{2} \int_0^{\hbar\omega_D/\Delta} \frac{ds}{\sqrt{1+s^2}} = \frac{1}{2} vg \sinh^{-1}\left(\frac{\hbar\omega_D}{\Delta}\right).$$

Zero temp gap

$$\Delta_0 \simeq 2\hbar\omega_D \, e^{-\frac{2}{g(\varepsilon_F) v}} .$$

[difference in argument of exponent as compared with the Cooper problem]. $\to e^{-\frac{4}{g(\varepsilon_F) v}}$. !

$g(\varepsilon_F) = 2N(0) \to$ for each spin

$\downarrow$

DOS for both spins

Justifies simple Cooper solution.

First application

$$\frac{E^S}{\cancel{\alpha}V} = \langle G | K^{MF}_{BCS} | G \rangle = \sum_k \left[\xi_k - E_k + \frac{|\Delta_k|^2}{2E_k} \right]$$

Subtract energy of metallic phase i.e. $\Delta_k = 0$.

$$E^M = \sum_{k < k_F} \xi_k \quad \left(\theta(k_F - k) \right) \qquad \text{[just the kinetic term]}$$

$$\frac{E^S - E^M}{V} = 2 \sum_k \left[(\xi_k - E_k) \, \theta(\xi_k) \, \theta(\hbar\omega_D - \xi_k) \right)$$

$$+ \sum_k \frac{\Delta_0^2}{2E_k} \, \theta(\hbar\omega_D - |\xi_k|).$$

(using nature of xn pot.)

$$= g(\varepsilon_F)\, \Delta_0^2 \int \ldots$$

$$\approx -\frac{1}{4} g(\varepsilon_F)\, \Delta_0^2 \equiv -\frac{B_c^2(0)}{2\mu_0}$$

$$\frac{B_c^2}{2\mu_0} = \frac{H_c^2}{8\pi}$$

$$\Rightarrow \quad B_c = \sqrt{\frac{\mu_0 \, g(\varepsilon_F)}{2}}\, \Delta_0.$$

$$B_c^2 = \frac{\mu_0}{4\pi} H_c^2.$$

$$B_c = \sqrt{\mu_0}$$

$$= V g(\varepsilon_F)\Delta_0^2 \int_0^{\hbar\omega_D/\Delta_0} ds \left[s - \sqrt{s^2+1} + \frac{1}{2\sqrt{s^2+1}} \right]$$

$$= V \frac{g}{2} \Delta_0^2 \left(x^2 - \sqrt{1+x^2} \right) \approx -\frac{1}{4} V g(\varepsilon_F) \Delta_0^2$$

$$x = \frac{\hbar\omega_D}{\Delta_0}$$

Number and phase

Consider a state:

$$|G(\alpha)\rangle = \prod_k \left[\cos\theta_k - e^{i\alpha} e^{i\phi_k} \sin\theta_k \, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\right] |0\rangle$$

$$\hat{N} |G(\alpha)\rangle = 2i \frac{\partial}{\partial\alpha} |G(\alpha)\rangle .$$

No: of Cooper pairs $\hat{M} = \frac{\hat{N}}{2}$ then $\hat{M} = i \frac{\partial}{\partial \alpha}$.

α & N are conjugates!

Projecting $|G(\alpha)\rangle$ onto states of definite particle number by defining

$$|M\rangle = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-iM\alpha} |G(\alpha)\rangle$$

State $|M\rangle$ has $N = 2M$ particles or N cooper pairs.

$$\frac{\langle G(\alpha)|N^2|G(\alpha)\rangle - \langle G(\alpha)|N|G(\alpha)\rangle^2}{\langle G(\alpha)|N|G(\alpha)\rangle} = 2\frac{\int d^3k \sin^2\theta_k \cos^2\theta_k}{\int d^3k \sin^2\theta} .$$

$\therefore \quad (\Delta N)_{RMS} \propto \sqrt{N}$

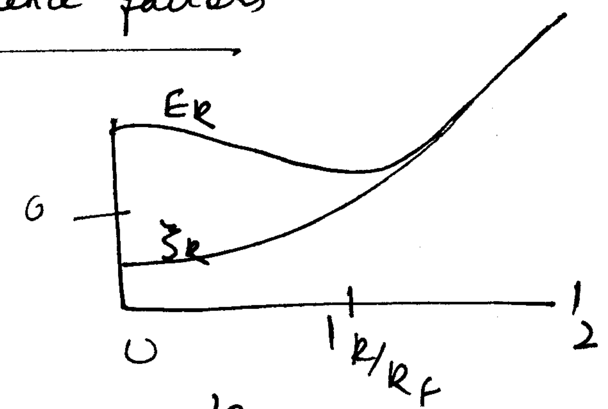
$\sin\theta_k = \theta(k_f - k)$
$\cos\theta_k = \theta(k - k_f)$
$= 0$ in fermi liquid regime

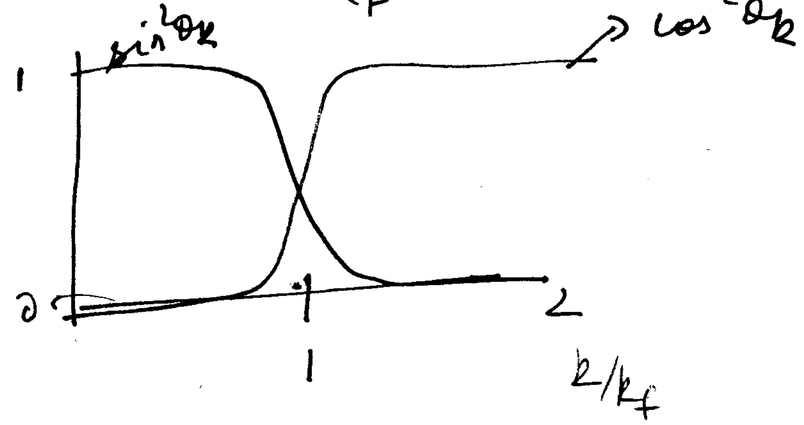
$\Delta\theta \, \Delta N \gtrsim 1$

In Normal phase, θ is indeterminate $\rightarrow$ U(1) ~~gauge~~ invariance

But in SC phase θ is determined due to U(1) symm breaking!

Coherence factors





$\sin^2\theta_k \to 1$ for $k < k_f$ width of variation window

$\cos^2\theta_k \to 1$ for $k > k_f$ $\delta k \simeq \frac{\Delta_0}{\hbar v_F} \to \xi^{-1}$ → coherence length.

Since $\gamma^+_{k\sigma} = \cos\theta_k c^+_{k\sigma} + r\sin\theta_k e^{-i\phi_k} c_{-k-\sigma}$

$\gamma^+_{k\sigma}$ creates e^- like exc^n in (k,σ) when $\cos\theta_k \to 1$

" hole like exc^n in $(-k,-\sigma)$ when $\sin\theta_k \to 1$

for $|k-k_f| \lesssim \frac{\Delta_0}{\hbar v_f}$, this operator creates a linear combo of e^- + hole states

Typically $\Delta_0 \sim 10^{-4} \varepsilon_f$ ε_f in metals ~~keV range~~. tens of thousands of Kelvin

$\delta k \lesssim 10^{-3} k_f$

Thus the entire physics of s wave SC states takes place within an onion skin at the fermi surface.

Finite temperature solution

Gap eqn: $$\Delta_k = - \sum_{k'} V_{kk'} \frac{\Delta_{k'}}{2E_{k'}} \tanh \frac{E_{k'}}{2k_B T}$$

clearly $\Delta_k \to 0$ as $T \to \infty$. $\sum_{k'} V_{kk'} \Delta_{k'} = -4k_B T \Delta_k$

If V_{kk} is bounded, no solution for $k_B T$ greater than largest eigenvalue of $V_{kk'}$

To find the critical temp, we recast this eqn.

$$1 = \frac{g(\varepsilon_F)}{2} V \int_0^{\hbar\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} \tanh \frac{\sqrt{\xi^2+\Delta^2}}{2k_B T}$$

T_c is when $\Delta \to 0$.

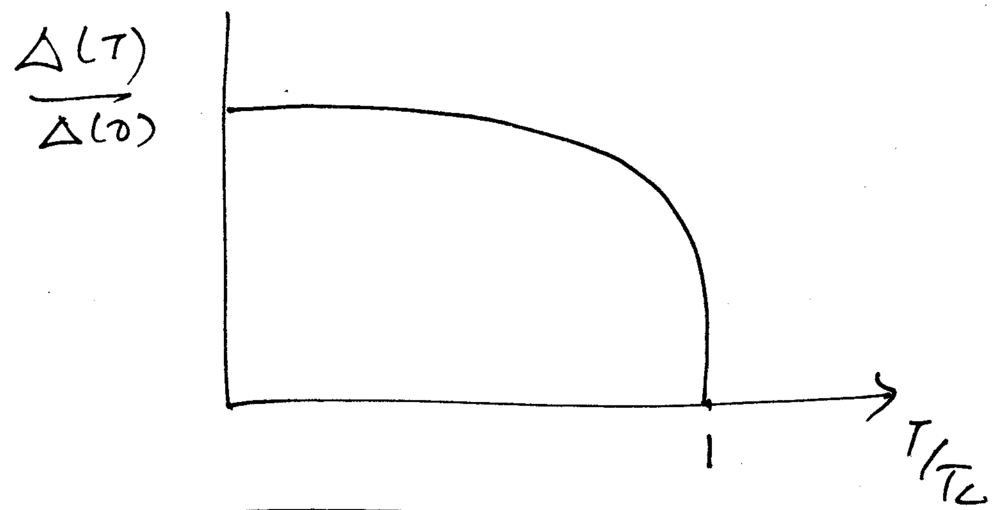
$$\frac{2}{g(\varepsilon_F) V} = \int_0^{\hbar\omega_D} \frac{\tanh \frac{\xi}{2k_B T_c}}{\xi} d\xi$$

$$\simeq \ln \frac{2\gamma \hbar\omega_D}{\pi k_B T_c}$$

$\gamma \to$ Euler const. $(0.577\ldots)$

$$\boxed{k_B T_c \simeq 1.13\, \hbar\omega_D\, e^{-\frac{2}{g(\varepsilon_F) V}}}$$

[



$$\Delta(0) = 1.76\, k_B T_c$$

we also know that

$$\Delta_0 \simeq 2\hbar\omega_D \exp -\frac{2}{g(\varepsilon_F)V}$$

$$k_B T_c = \frac{1.13}{2}\Delta(0).$$

$$\Delta(0) \simeq 1.76\, k_B T_c$$

$2\Delta \approx 3.5\, T_c$. BCS relation

Old s wave SC $\rightarrow 3.0 < \frac{2\Delta}{T_c} < 4.5\, T_c$. !

(no true for unconventional SC !

Below Tc $\quad T \to T_c^-$

$$\Delta(T) \approx 3.06\, k_B T_c \left[1 - \frac{T}{T_c}\right]^{1/2}.$$

Isotope effect

Things are proportional to $\hbar\omega_D$.

$$\therefore \ln T_c = \ln \omega_D - \frac{2}{g(\varepsilon_F) v} + \text{const}.$$

$\omega \sim \sqrt{\frac{k}{m}}$.

If we can vary the mass of ions) via isotopic substitution while not changing $g(\varepsilon_F)$ & v

then $\delta \ln T_c = \delta \ln \omega_D = -\frac{1}{2} \delta \ln M$.

$\therefore$ increasing M decreases T_c!

What is the impact of repulsive x^{ns}?

$$V_{kk'} = \begin{cases} v_c - v_p / V & |\xi_k|, |\xi_{k'}| < \hbar\omega_D \\ v_c / V & \text{otherwise.} \end{cases}$$

$$\Delta_k = \begin{cases} \Delta_0 \text{ real} & |\xi_k| < \hbar\omega_D \\ \Delta_1 \text{ - real otherwise.} \end{cases}$$

Assume $v_p > v_c$ to have any attraction!
(v_p → phonons, v_c → coulomb repulsion)

The gap equation. leads to 2 eqns. for Δ_0 & Δ_1 now.

$$k_B T_c = 1.134\, \hbar\omega_D \exp - \frac{2}{g(\varepsilon_F) v_{eff}}$$

$$v_{eff} = v_p - \frac{v_c}{1 + \frac{1}{2} g(\varepsilon_F) v_c \ln B/\hbar\omega_D}$$

B – bandwidth of electrons

At $T=0$, gap equation,

$$\Delta_0 = \frac{1}{2} g(\varepsilon_F)(v_p - v_c)\int_0^{\hbar\omega_D} d\xi \frac{\Delta_0}{\sqrt{\xi^2+\Delta_0^2}} - \frac{1}{2} g(\varepsilon_F) v_c \int_{\hbar\omega_D}^{B} \frac{\Delta_1}{\sqrt{\xi^2+\Delta_1^2}}$$

$$\Delta_1 = -\frac{1}{2} g(\varepsilon_F) v_c \int_0^{\hbar\omega_D} \checkmark \quad - \frac{1}{2} g(\varepsilon_F) v_c \int \checkmark$$

$\omega_D \to$ Debye freq

$B \to$ Bandwidth

assume $\Delta_{0,1} \ll \hbar\omega_D \ll B$

$$\Delta_1 = -\frac{\frac{1}{2} g v_c \ln\left(\frac{2\hbar\omega_D}{\Delta_0}\right)}{1+\frac{1}{2} g v_c \ln B/\hbar\omega_D}\Delta_0$$

Soln. only if

$$\frac{2}{g v_p} = \frac{1}{v_p}\ln\frac{2\hbar\omega_D}{\Delta_0}\left[v_p - v_c \cdot \frac{1}{1+\frac{g}{2}\ln B/\hbar\omega_D\, v_c}\right]$$

$B \gg \hbar\omega_p$. $e^{\frac{1}{x}}$ $e^{-\frac{1}{x}}$

$e^{-\frac{1}{\cdots}}$ $V_{eff} \downarrow$ & $T_c \downarrow$. $e^{\frac{1}{1/x}}$.

T_c decreases with repulsive x^{ns}.

& so does Δ_0.

$\lambda = \frac{g}{2} \nu_p$ $\mu = \frac{g}{2} \nu_c$ $\mu^* = \frac{\mu}{1+\mu \ln B/\hbar\omega_p}$

then $k_B T_c = 1.134\, \hbar\omega_p\, e^{-1/\lambda-\mu^*}$

Isotope effect is also modified.