
Exercise 10.1 Fidelity and Uhlmann's Theorem

Given two states ρ and σ on $\mathcal{H}_A$ with fixed basis $\{|A\rangle_i\}_i$ and a reference Hilbert space $\mathcal{H}_B$ with fixed basis $\{|B\rangle_i\}_i$, which is a copy of $\mathcal{H}_A$, Uhlmann's theorem claims that the fidelity can be written as

$$F(\rho,\sigma) = \max_{|\psi\rangle,|\phi\rangle} |\langle\psi|\phi\rangle|\,, \tag{1}$$

where the maximum is over all purifications $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ on $\mathcal{H}_A \otimes \mathcal{H}_B$. Let us introduce a state $|\psi\rangle$ as:

$$|\psi\rangle = (\sqrt{\rho} \otimes U_B)\,|\gamma\rangle, \qquad |\gamma\rangle = \sum_i |A\rangle_i \otimes |B\rangle_i, \tag{2}$$

where U_B is any unitary on $\mathcal{H}_B$.

a) *Show that $|\psi\rangle$ is a purification of ρ.*

We need to show that $|\psi\rangle$ is normalized and that it reduces to ρ when we trace out $\mathcal{H}_B$. The latter holds since

$$\begin{aligned}
\mathrm{Tr}_B(|\psi\rangle\langle\psi|) &= \mathrm{Tr}_B\big(\sum_{i,j} \sqrt{\rho}|i\rangle\langle j|_A\sqrt{\rho} \otimes U_B|i\rangle\langle j|_B U_B^\dagger\big)\\
&= \sum_m \sqrt{\rho}|m\rangle\langle m|_A\sqrt{\rho}\\
&= \rho.
\end{aligned}$$

Normalization follows from $\mathrm{Tr}|\psi\rangle\langle\psi| = \mathrm{Tr}\rho = 1$.

b) *Argue why every purification of ρ can be written in this form.*

We have seen previously that all purifications are equivalent up to a unitary transformation on $\mathcal{H}_B$. The proposition directly follows from this, since U_B can be chosen arbitrarily.

c) *Use the construction presented in the proof of Uhlmann's theorem to calculate the fidelity between $\sigma' = \mathbb{1}_2/2$ and $\rho' = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ in the 2-dimensional Hilbert space with computational basis.*

It is sufficient to maximize over one set of purifications. We set

$$\begin{aligned}
|\psi\rangle &= (\sqrt{\rho'} \otimes V_B)|\gamma\rangle\\
|\phi\rangle &= \frac{1}{\sqrt{2}}(\mathbb{1}_A \otimes \mathbb{1}_B)|\gamma\rangle \quad = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).
\end{aligned}$$

It follows that

$$\begin{aligned}
|\langle\psi|\phi\rangle| &= \frac{1}{\sqrt{2}}\left|\langle\gamma|\sqrt{\rho'} \otimes V_B|\gamma\rangle\right|\\
&= \frac{1}{\sqrt{2}}\left|\mathrm{Tr}\left(\sqrt{\rho'}\cdot V_B^T\right)\right|\\
&\le \frac{1}{\sqrt{2}}\mathrm{Tr}\left(\left|\sqrt{\rho'}\right|\right)\\
&= \frac{1}{\sqrt{2}}\left(\sqrt{p} + \sqrt{1-p}\right).
\end{aligned}$$

We used Lemma 9.5 and the result of Exercise 9.16 from Nielson/Chuang (pages 410-411) to derive this result.

This maximum can be achieved when V_B^T produces the polar decomposition of $\sqrt{\rho'}$ — which in this case is trivially $V_B = \mathbb{1}_B$. We obtain $F(\rho', \sigma') = (\sqrt{p} + \sqrt{1-p})/\sqrt{2}$.

d) *Give an expression for the fidelity between any pure state and the completely mixed state $\mathbb{1}_n/n$ in the n-dimensional Hilbert space.*

The general case follows immediately from the original definition:

$$
\begin{aligned}
F(\rho, \sigma) &= \operatorname{Tr}\left(\sqrt{\sqrt{\sigma'}\rho'\sqrt{\sigma'}}\right) \\
&= \frac{1}{\sqrt{n}}\operatorname{Tr}\left(\sqrt{\rho'}\right) \\
&= \frac{1}{\sqrt{n}} \; .
\end{aligned}
$$

The last equality follows from the fact that $\sqrt{\rho'} = \rho'$ for pure states.

Exercise 10.2 Resource inequalities

By assumption, messages to be sent are chosen uniformly at random and we have a perfect super dense coding. Hence $I(X : X') = 2n$, where X is classical input, and X' classical guess. Now by Holevo bound we know that mutual information between input X and Bob's quantum guess is greater than that between the classical input and a guess, hence: $I(X : B) \geq I(X : X')$ and $I(X : B) = H(B) - H(B|X) \leq H(B) \leq n(\alpha + \beta)$ Hence we have that $2n \leq n(\alpha + \beta)$, hence $\alpha + \beta \geq 2$.