Quantum Information Theory
Series 6

HS 14
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Exercise 6.1 Classical capacity of the depolarizing channel

Consider the depolarizing channel we have treated before, that is described by the CPTP map:

$$E_p : \mathcal{S}(\mathcal{H}_A) \mapsto \mathcal{S}(\mathcal{H}_B)$$
$$\rho \to p\frac{\mathbb{1}}{2} + (1-p)\rho.$$

a) Now we will see what happens when we use this quantum channel to send classical information. We start with an arbitrary input probability distribution $P_X(0) = q, P_X(1) = 1-q$. We encode this distribution in a state $\rho_X = q\,|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$. Now we send ρ_X over the quantum channel, i.e., we let it evolve under $E_\mathbf{p}$. Finally, we measure the output state, $\rho_Y = E_\mathbf{p}(\rho_X)$ in the computational basis.

Compute the conditional probabilities $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

b) Maximize the mutual information over q to find the classical channel capacity of the depolarizing channel.

c) What happens to the channel capacity if we measure the final state in a different basis?

Exercise 6.2 A sufficient entanglement criterion

In general it is very hard to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that works fine at least in low dimensions.

a) Show that the *transpose* is a positive operation, and that it is basis-dependent.

b) Let $\rho \in End(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a separable state, and let Λ_A be a positive operator on $\mathcal{H}_A$. Show that $\Lambda_A \otimes \mathbb{1}_B$ maps ρ to a positive operator.

The task of characterizing the sets of separable states then reduces to finding a suitable positive map that distinguishes between separable and entangled states.

c) Show that the *transpose* is a probable candidate by testing it on a Werner state (impure singlet)

$$W = x|\psi^-\rangle\langle\psi^-| + (1-x)\,\mathbb{1}/4,$$

where $x \in [0,1]$ and $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. What happens to the eigenvalues of W when you apply *transpose* $\otimes\, \mathbb{1}_B$?

Remark: Indeed, it can be shown that the PPT (*positive partial transpose*) criterion is necessary and sufficient for systems of dimension 2×2 and 2×3.

d) Show that although the partial transpose is basis-dependent, the corresponding eigenvalues are independent under local basis-transformations.

Exercise 6.3 Uncertainty Principle

Let ρ be a density operator, A and B observables, and $t \in R$. Show that

- $(A + itB)\rho(A + itB)^*$ is positive (Alternatively, show directly that $\langle (A + itB)(A + itB)^* \rangle$ is positive)
- $4\langle A^2 \rangle_\rho \langle B^2 \rangle_\rho \geq |\langle [A, B] \rangle_\rho|^2$

Hence deduce the Uncertainty Principle $\Delta_\rho[P]^2 \Delta_\rho[X]^2 \geq \frac{1}{4}\hbar^2$.

Exercise 6.4 Stinespring Isometry

(Extra question) Show the existence of the Stinespring isometry directly from the Choi state.