Quantum Information Theory
Series 3

HS 14
Dr. J.M. Renes

Exercise 3.1 Reduced density matrix - partial trace

Partial trace is an important concept in the quantum mechanical treatment of multi-partite systems, and is the natural generalisation of the concept of marginal distributions in classical probability theory.
Assume two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with orthonormal bases $\{\xi_j : j = 1, ..., m\}$ and $\{\eta_k : k = 1, ..., n\}$, respectively, and vector $|\Psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ given by

$$|\Psi\rangle_{AB} = \sum_{j,k} C_{j,k} |\xi_j\rangle |\eta_k\rangle$$

Reduced density matrix of a system A is defined via a partial trace on the whole system:

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_{k=1}^{n} \langle \eta_k | \rho_{AB} | \eta_k \rangle \tag{1}$$

a) Show that the reduced state on a system A can be written as:

$$\rho_A = \sum_{j,k,r} C_{jk} \overline{C}_{rk} |\xi_j\rangle\langle\xi_r|$$

and deduce that the matrix of ρ_A with respect to the basis $\{\xi_j\}$ can be written as $CC^\dagger$, where C is the $m \times n$ matrix with entries C_{jk}, and $C^\dagger$ its transpose. Also show that the matrix of ρ_B is $C^\dagger C$. Deduce that ρ_A and ρ_B must have same non-negative eigenvalues.

b) Show that ρ_A is a valid density operator by proving it is:

1) Hermitian: $\rho_A = \rho_A^\dagger$.
2) Positive: $\rho_A \geq 0$.
3) Normalised: $\mathrm{Tr}(\rho_A) = 1$.

c) Find ρ_A and ρ_B in the case when H_A and H_B have orthonormal bases $\{v_0, v_1, v_2\}$ and $\{\omega_1, \omega_2\}$, respectively (hence $m = 3, n = 2$), and the (unnormalised) state ψ is given by

$$|\Psi\rangle_{AB} = |v_0\rangle(|\omega_1\rangle - |\omega_2\rangle) + |v_1\rangle|\omega_1\rangle + |v_2\rangle|\omega_2\rangle$$

Show that ρ_A and ρ_B have the same non-zero eigenvalues.

d) Calculate the reduced density matrix of the system A in the Bell state

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right).$$

e) Consider a classical probability distribution P_{XY} with marginals P_X and P_Y.

1) Calculate the marginal distribution P_X for

$$P_{XY}(x,y) = \begin{cases} 0.5 & \text{for } (x,y) = (0,0), \\ 0.5 & \text{for } (x,y) = (1,1), \\ 0 & \text{else,} \end{cases}$$

with alphabets $\mathcal{X}, \mathcal{Y} = \{0,1\}$.

2) How can we represent P_{XY} in form of a quantum state?

3) Calculate the partial trace of P_{XY} in its quantum representation.

f) Can you think of an experiment to distinguish the bipartite states of parts *d*) and *e*)?

Exercise 3.2 State Distinguishability

One way to understand the cryptographic abilities of quantum mechanics is from the fact that non-orthogonal states cannot be perfectly distinguished.

a) In the course of a quantum key distribution protocol, suppose that Alice randomly chooses one of the following two states and transmits it to Bob:

$$|\phi_0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad \text{or} \qquad |\phi_1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + i|1\rangle).$$

Eve intercepts the qubit and performs a measurement to identify the state. The measurement consists of the orthogonal states $|\psi_0\rangle$ and $|\psi_1\rangle$, and Eve guesses the transmitted state was $|\phi_0\rangle$ when she obtains the outcome $|\psi_0\rangle$, and similar for $|\phi_1\rangle$. What is the probability that Eve correctly guesses the state, averaged over Alice's choice of the state for a given measurement? What is the optimal measurement Eve should make (choice of $|\psi_0\rangle$), and what is the resulting optimal guessing probability?

b) Now suppose Alice randomly chooses between two states separated by an angle θ on the Bloch sphere. What is the measurement which optimizes the guessing probability? What is the resulting probability of correctly identifying the state?

Exercise 3.3 One-qubit POVM

Consider a single qubit and unit vectors $\vec{n}_k, k \in \{1, ..., n\}$ such that

$$\sum_k \lambda_k \vec{n_k} = 0$$

for $\lambda_k \in (0,1)$ and $\sum_k \lambda_k = 1$. Show that a measurement on a qubit defined by

$$F_k = 2\lambda_k | \uparrow_{\vec{n_k}}\rangle\langle\uparrow_{\vec{n_k}} |$$

is a POVM. Explain cases $N = 2$ and $N = 3$, and connect them to the Bloch sphere representation. For the case $N = 3$ think of suitable vectors $\vec{n_k}$ and extend above POVM measurement on a qubit to the orthogonal measurement on a qutrit in a suitable basis (*Neumark's theorem is a generalisation of this fact*).