ETH
Eidgenössische Technische Hochschule Zürich
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Quantum Information Theory
Series 1

HS 14
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Exercise 1.1 Trace distance

The trace distance (or L_1-distance) between two probability distributions P_X and Q_X over a discrete alphabet $\mathcal{X}$ is defined as

$$\delta(P_X, Q_X) = \frac{1}{2} \sum_{x \in \mathcal{X}} |P_X(x) - Q_X(x)| \,. \tag{1}$$

The trace distance may also be written as

$$\delta(P_X, Q_X) = \max_{\mathcal{S} \subseteq \mathcal{X}} |P_X[\mathcal{S}] - Q_X[\mathcal{S}]| \,, \tag{2}$$

where we maximise over all events $S \subseteq \mathcal{X}$ and the probability of an event is given by $P_X[\mathcal{S}] = \sum_{x \in \mathcal{S}} P_X(x)$.

a) Show that $\delta(\cdot, \cdot)$ is a good measure of distance by proving that $0 \leq \delta(P_X, Q_X) \leq 1$ and the triangle inequality $\delta(P_X, R_X) \leq \delta(P_X, Q_X) + \delta(Q_X, R_X)$ for arbitrary probability distributions P_X, Q_X and R_X.

b) Show that definitions (2) and (1) are equivalent.

c) Let us now find an operational meaning for the trace distance. Suppose that P_X and Q_X represent the probability distributions of the outcomes of two dice that look identical. You are allowed to throw one of them only once and then have to guess which die that was. What is your best strategy? What is the probability that you guess correctly and how can you relate that to the trace distance $\delta(P_X, Q_X)$?

Exercise 1.2 Weak law of large numbers

Let A be a positive random variable with expectation value $\langle A \rangle = \sum_a a \; P_A(a)$. Let $P[A \geq \varepsilon]$ denote the probability of an event $\{A \geq \varepsilon\}$, for $\varepsilon > 0$.

a) Prove Markov's inequality

$$P[A \geq \varepsilon] \leq \frac{\langle A \rangle}{\varepsilon}. \tag{3}$$

b) Use Markov's inequality to prove Chebyshev's inequality, *i.e.*,

$$P\left[(X - \mu)^2 \geq \epsilon\right] \leq \frac{\sigma^2}{\epsilon}, \tag{4}$$

where $\mu = \langle X \rangle$ and σ denotes the standard deviation of X.

c) Now use Chebyshev's inequality to prove the weak law of large numbers for i.i.d. random variables X_i:

$$\lim_{n \to \infty} P\left[\left(\frac{1}{n}\sum_i^n X_i - \mu\right)^2 \geq \varepsilon\right] = 0 \quad \text{for any } \varepsilon > 0 \text{ and } \mu = \langle X_i \rangle, \forall i \in \{1, ..., n\}. \tag{5}$$

Exercise 1.3 Jensen's inequality

Given a convex function f, and the probability distribution $\{p_1, .., p_n\}$, prove Jensen's inequality:

$$f(\sum_{k=1}^{n} p_k x_k) \leq \sum_{k=1}^{n} p_k f(x_k)$$

In the language of Probability theory, with ϕ a convex function, and X a random variable, this inequality can be written as:

$$\phi[E(X)] \leq E[\phi(X)]$$

Exercise 1.4 Conditional probabilities: how knowing more does not always help

Suppose you are visiting your grandfather in his hut in Scotland. You had offered him a radio for Christmas three years ago, but he is not fond of such modern technologies and has not used it since. You decide to initiate a game to prove to him that technology is helpful: every evening you alone listen to the weather forecast on the radio and then both you and your grandfather try to guess if it will rain next morning. Having lived there since birth, your grandfather knows that it rains on 80% of the days. You had reached the same conclusion on previous summer holidays. You also know that the weather forecast is right 80% of the time and is always correct when it predicts rain.

a) What is the optimal strategy for your grandfather? And for you?

b) Both of you keep a record of your guesses and the actual weather for statistical analysis. After some time (*i.e.* enough that you can apply the weak law of large numbers), who will have guessed correctly more often?