# Short Introduction To Special Relativity
# Lecture Notes*

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**Inertial coordinate system:** A coordinate system in which free particles (in absence of forces) satisfy the equation of motion

$$\ddot{x} = 0 \tag{1}$$

is called inertial coordinate system. In particular: two inertial coordinate systems move with constant velocity to each other.

**Postulates of SR:**

1. The laws of nature are independent of the choice of coordinate system. In particular: any formula describing them has to have the same form in all inertial systems.
2. The speed of light is the same in all coordinate systems

**Events** Events in $\mathbb{R}^{1+3}$ space-time are 4-vectors

$$X = \left(X^0, \quad X^1, \quad X^2, \quad X^3\right) = \left(ct, \underbrace{x, \ y, \ z}_{\vec{x}}\right). \tag{2}$$

One refers to components of a 4-vector by $X^\mu, \ \mu \in \{0, 1, 2, 3\}$. Often one is interested in the space-time separation of two events $\Delta X = X_1 - X_2$.

**Metric** We define the metric

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

The metric allows to quantify space-time 'distance$^2$'

$$(\Delta X)^2 := \eta_{\mu\nu} \Delta X^\mu \Delta X^\nu = (c\Delta t)^2 - (\Delta \vec{x})^2 \tag{4}$$

Since the metric is indefinite there are three cases. We define:

- $\Delta X$ is space-like if $(\Delta X)^2 < 0$
- $\Delta X$ is light-like if $(\Delta X)^2 = 0$
- $\Delta X$ is time-like if $(\Delta X)^2 > 0$

*based on the relevant chapters of the lecture notes [Graf, Renner]

**Transformation between inertial frames** 1st postulate $\Rightarrow$ Free particle move in all inertial coordinate systems on a straight line $\Rightarrow X^\mu \mapsto A^\mu_{\ \nu} X^\nu + a^\mu$. Coordinate differences transform then as

$$\Delta X^\mu \mapsto A^\mu_{\ \nu} \Delta X^\nu. \tag{5}$$

2nd postulate $\Rightarrow$ light-like distances have to be light-like in all coordinate systems: $\Delta X^\nu A^\mu_{\ \nu} \eta_{\mu\sigma} A^\sigma_{\ \rho} \Delta X^\rho = 0$ for $\Delta X$ light-like. One can show that this requirement leads to

$$A^\mu_{\ \nu} \eta_{\mu\sigma} A^\sigma_{\ \rho} = \alpha^2 \eta_{\nu\rho},\ \alpha \in \mathbb{R}. \tag{6}$$

We can write $A = \alpha\Lambda$ which defines an element $\Lambda$ in the Lorentz group $L$:

**Definition:** The Lorentz group $L$ is defined by the linear transformations $\Lambda$ that leave the metric invariant

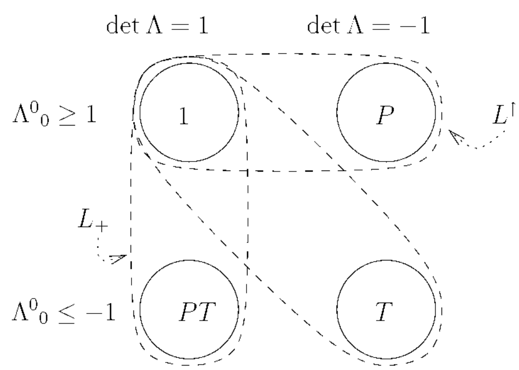
$$\Lambda^\mu_{\ \nu} \eta_{\mu\sigma} \Lambda^\sigma_{\ \rho} = \eta_{\nu\rho}. \tag{7}$$

**Properties:** Taking the determinant and the $\nu = 0, \mu = 0$ component of (7) leads to

$$(\det(L))^2 = 1 \tag{8}$$

$$1 = \Lambda^\mu_{\ 0} \eta_{\mu\sigma} \Lambda^\sigma_{\ 0} = \Lambda^0_{\ 0}\Lambda^0_{\ 0} - \sum_{i=1}^{3} \Lambda^i_{\ 0}\Lambda^i_{\ 0} \quad \Rightarrow \Lambda^0_{\ 0} \geq 1 \vee \Lambda^0_{\ 0} \leq -1 \tag{9}$$

Thus the Lorentz group has four connected components:




Examples for each component are the reflections

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$PT = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The transformations $L^\uparrow_+ = \left\{\Lambda \in L \middle| \det\Lambda = 1,\ \Lambda^0_{\ 0} \geq 1\right\}$ form a sub group: the **proper orthochronous Lorentz-transformations**. Any general element in $L$ can be written as an element in $L^\uparrow_+$ times one of the reflections.

**Examples:**

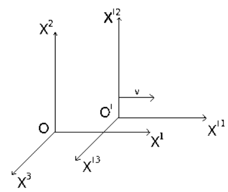
- Spatial rotations

$$\Lambda(R) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix},\ R \in SO(3)$$

- Boost in $x^1$-direction

$$\Lambda(v) := \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \geq 1$$

  Transforms coordinates from the reference frame $\mathcal{O}$ to the reference frame $\mathcal{O}'$, with aligned spatial axes, moving with constant velocity $v$ in $x^1$-direction.




A general element $\Lambda \in L_+^\uparrow$ can be written as $\Lambda = \Lambda(R_1)\Lambda(v)\Lambda(R_2)$.

**Example** Let us write down the boost for each component

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \gamma\left(c\Delta t - \frac{v}{c}\Delta x\right) \\ \gamma\left(-v\Delta t + \Delta x\right) \\ \Delta y \\ \Delta z \end{pmatrix}$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right), \tag{10}$$

$$\Delta x' = \gamma\left(-v\Delta t + \Delta x\right). \tag{11}$$

note: The coordinates perpendicular to $\vec{v}$ are not affect by the boost.

**Time dilation:** Let us consider a clock in its rest frame $\mathcal{O}$. We consider the time difference $\Delta t$ between two ticks, since the clock does not move: $\Delta x = 0$. In the frame $\mathcal{O}'$ of some observer passing the clock with velocity $v$ we find

$$\Delta t' = \gamma\Delta t \geq \Delta t \tag{12}$$

$$\Delta x' = \gamma\left(-v\Delta t\right) \tag{13}$$

The passing observer measures with a clock of his reference frame $\mathcal{O}'$ a longer time interval $\Delta t'$ between two ticks of the clock in $\mathcal{O}$ than the clock in $\mathcal{O}$ itself. Note that seen from $\mathcal{O}'$ the clock in $\mathcal{O}$ has moved between the two ticks by $\Delta x'$.

- The time interval $\Delta t$ measured by a clock at a fixed location is called **proper time**. In any other reference frame $\Delta t' \geq \Delta t$.

**Simultaneity:** Let us consider two events $X_{1/2} = \left(ct_{1/2}, \vec{x}_{1/2}\right)$, e.g. two flashes, that happen at the same time in $\mathcal{O}$, i.e. $\Delta t = t_1 - t_2 = 0$. In a reference frame $\mathcal{O}'$ moving passed with velocity $v$ we observe

$$\Delta t' = \gamma\left(-\frac{v}{c^2}\Delta x\right), \tag{14}$$

$$\Delta x' = \gamma\left(\Delta x\right). \tag{15}$$

Thus the two events do not happen at the same time in $\mathcal{O}'$, $\Delta t' \neq 0$. Simultaneity depends on the reference frame.

**Length contraction:** Let us consider an object of length $\Delta x = l$ in its rest frame $\mathcal{O}$. In order to determine the length of an object we determine the coordinates of its endpoints $X_{1/2} = \left(ct_{1/2}, \vec{x}_{1/2}\right)$. In the rest frame of the object it doesn't matter at what time we measure the endpoints, since the object is at rest. However in order to measure the length in a reference frame $\mathcal{O}'$ passing with $v$, we have to determine the end points $X'_{1/2}$ at the same time, i.e. $\Delta t' = 0$, since the object is moving w.r.t $\mathcal{O}'$.

$$
\begin{aligned}
0 &= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \\
l' = \Delta x' &= \gamma\left(1 - \frac{v^2}{c^2}\right)\Delta x \\
\Rightarrow l' &= \frac{l}{\gamma} \leq l
\end{aligned}
\tag{16}
$$

Hence the object appears shorter in $\mathcal{O}'$.

- The length $l$ of an object measured in a reference frame $\mathcal{O}$ where the object is at rest is caller **proper length**. In any other reference frame $\mathcal{O}'$, $l' \leq l$.

**Addition of velocities** Consider an object moving with velocity $u'$ measured in coordinate system $\mathcal{O}'$, which is moving with velocity $v$ with respect to coordinate system $\mathcal{O}$. What is the velocity $u$ of the object measured in $\mathcal{O}$?

In $\mathcal{O}'$ the object covers a distance $dx' = u'\, dt'$ in a time interval $dt'$. Transformed to the coordinate system $\mathcal{O}$ (now we have to use $\Lambda(v)^{-1} = \Lambda(-v)$ ) we get

$$
dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right), \tag{17}
$$

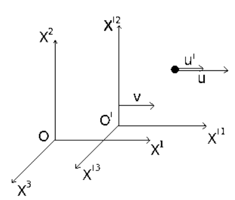
$$
dx = \gamma\left(vdt' + dx'\right). \tag{18}
$$

Hence

$$
u = \frac{dx}{dt} = \frac{\left(v + \frac{dx'}{dt'}\right)}{\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} = \frac{(v + u')}{\left(1 + \frac{vu'}{c^2}\right)} \leq (v + u'). \tag{19}
$$

In particular

$$
u', v \leq c \;\Rightarrow u \leq c. \tag{20}
$$




**World-line and Action** A moving point particle traces out a line in Minkowski space $X(t) = \left(ct, \quad \vec{x}(t)\right)$ called world-line. We would like to write down an action $A$ that is both reparametrization- and Lorentz invariant:

$$
A[X] \propto \int \sqrt{\eta_{\mu\nu}\dot{X}^\mu(t)\dot{X}^\nu(t)}dt = \int \sqrt{c^2 - v^2}dt = c\int \frac{1}{\gamma}dt = c\int d\tau \tag{21}
$$

where $\tau$ is the proper time of the particle, i.e. the time measured in its rest frame. The units of an action should be $[energy] \times [time]$. Therefore the proportionality constant has to have units $[mass] \times [velocity]$. Furthermore the constant has to be a Lorentz invariant quantity. The obvious choice is $mc$, where $m$ is the Lorentz invariant rest mass of the particle. Thus the complete action reads

$$
A = \int L\, dt, \qquad\qquad L = mc\sqrt{\eta_{\mu\nu}\dot{X}^\mu(t)\dot{X}^\nu(t)}. \tag{22}
$$

**Energy and momentum conservation** As in classical mechanics, the canonical momentum and the equations of motion are given by

$$p^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = \frac{mc\dot{X}^\mu}{\sqrt{\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}} = \gamma m \underbrace{\dot{X}^\mu}_{(c,\,\vec{v}(t))}, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{X}_\mu} - \underbrace{\frac{\partial L}{\partial X_\mu}}_{=0} = \frac{dp^\mu}{dt} = 0. \tag{23}$$

In fact the equations of motion are stating that energy and momentum are conserved for a free particle. The conserved quantity associated with time translation invariance is the energy. The $p^0$ component of the momentum is the conserved quantity associated to invariance of $x^0 = ct$. Therefore we interpret

$$E = cp^0 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{24}$$

as relativistic energy. In the limit $v \ll c$

$$E \approx mc^2 + \frac{1}{2}mv^2 + \cdots \tag{25}$$

we recover the classical expression for the kinetic energy $\frac{1}{2}mv^2$, but also a new term $mc^2$, which is also present when the particle is at rest. We interpret the latter as the **rest energy** of the particle.

**4-velocity and 4- momentum** Instead of parameterizing the world-line $X(t) = (ct,\ \vec{x}(t))$ with the time of the observing reference frame, we can use the proper time $\tau$ of the particle to parameterize the world-line: $X(\tau) := \big(ct(\tau),\quad \vec{x}(t(\tau))\big)$. We define the 4-velocity as

$$u^\mu = \frac{dX^\mu}{d\tau}. \tag{26}$$

The 4-momentum found above can then be written in terms of the 4-velocity as

$$p^\mu = mu^\mu. \tag{27}$$

They transform as vectors for orthochronous Lorentz transformation. For general Lorentz transformations they pick up the sign $\mathrm{sgn}(\Lambda^0_{\ 0})$, and hence transform as pseudo vectors (due to the choice of the positive root in the definition of the proper time $d\tau = \sqrt{1-\frac{v^2}{c^2}}dt$).

In the rest-frame $\mathcal{O}'$ of the particle we have $u' = (c, 0, 0, 0)$, thus $\eta_{\mu\nu}u'^\mu u'^\nu = c^2$. Since the metric is Lorentz invariant, $u^\mu u_\mu = c^2$ in any reference frame and similarly $p^\mu p_\mu = m^2c^2$. Hence using $E = cp^0$

$$\begin{aligned}\left(p^0\right)^2 - \vec{p}^2 &= m^2c^2,\\ E &= \sqrt{m^2c^4 + \vec{p}^2c^2}.\end{aligned} \tag{28}$$

The last equation is also valid for massless particles. Then $E = c|\vec{p}|$ and the 4-momentum $\big(|\vec{p}|,\quad \vec{p}\big)$.

**Example: Decay of particle** Let a particle of rest mass $M$ decay symmetrically into two particles, each of rest mass $m$. In the rest frame of the initial particle we have the 4-momentum $P^\mu = \big(cM,\quad 0,\quad 0,\quad 0\big)$. After the decay the two particles have 4-momentum $p^\mu_\pm = \gamma\left(cm,\quad \pm m\vec{v}\right)$ due to conservation of the $P^i$, $i \in \{1,2,3\}$ components of the 4-momentum . The conservation of $P^0$ yields

$$cM = 2\gamma cm, \qquad \Rightarrow\ 2m = \frac{M}{\gamma} < M. \tag{29}$$

The **total mass is not conserved**, some of the rest energy was transformed into kinetic energy: for each particle $E_{kin} = E - mc^2 = \frac{1}{2}Mc^2\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)$.

**Electrodynamics** There are two unit systems that are frequently used in electrodynamics, SI and cgs. We give the relevant expressions in both systems. The electromagnetic fields $\vec{E}$ and $\vec{B}$ can be described in terms of the scalar potential $\phi$ and the vector potential $\vec{A}$. In order to find a relativistic covariant description of electrodynamics we combine both potentials to a 4-potential and define it as 1-form

$$\text{SI}: \; A = \big(A_0, \quad A_1, \quad A_2, \quad A_3\big) = \big(\tfrac{\phi}{c}, \quad -\vec{A}\big), \qquad\qquad \text{cgs}: A = \Big(\phi, -\vec{A}\Big). \tag{30}$$

The electromagnetic field tensor is then defined as the exterior derivative d of $A$. Those who are unfamiliar with exterior derivative may content themselves with the explicit definition

$$F = \mathrm{d}A, \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{31}$$

In components, the field tensor takes following form

$$\text{SI}: \; (F_{\mu\nu}) = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}, \qquad \text{cgs}: \; (F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{32}$$

Electromagnetic fields are created due to the presence of electric charge density $\rho$ and current density $\vec{j}$. The charge density $\rho_0$ measured in the rest frame of the charges is perceived in a frame moving with velocity $\vec{v}$, as

$$\rho = \gamma\rho_0. \tag{33}$$

This follows from the fact that the volume element is length contracted in one direction. Given the charge density $\rho$ the current density is as usual

$$\vec{j} = \rho\vec{v}. \tag{34}$$

We also need to express the sources in a covariant way. We define the 4-current density

$$j^\mu = \big(c\rho, \quad \vec{j}\big) = \rho_0 u^\mu. \tag{35}$$

The continuity equation takes the very simple and Lorentz invariant form

$$\partial_\mu j^\mu = 0 \tag{36}$$

since explicitly $\partial_\mu j^\mu = \frac{\partial\rho}{\partial t} + \mathrm{div}\vec{j}$.

**Maxwell equations** The inhomogeneous Maxwell equation are expressed as

$$\text{SI}: \; \partial_\mu F^{\mu\nu} = \mu_0 j^\nu. \qquad\qquad \text{cgs}: \partial_\mu F^{\mu\nu} = \frac{j^\nu}{c}. \tag{37}$$

The homogeneous Maxwell equations follow immediately from the very definition of the field tensor and the property of the exterior derivative $\mathrm{d}\circ\mathrm{d} = 0$. Again, those unfamiliar with the exterior derivative may be satisfied with the explicit expression.

$$\mathrm{d}F = \mathrm{d}\,(\mathrm{d}A) = 0, \qquad\qquad (\mathrm{d}F)_{\mu\nu\rho} = \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{38}$$

**Example: inhom. M. eq.**$\nu = 1$ We do this example only in SI units. First we need to lift the indices of the field tensor

$$F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}, \tag{39}$$

where $(\eta^{\mu\nu}) = (\eta_{\mu\nu})^{-1}$ which happens to have the same matrix entries as $(\eta_{\mu\nu})$. Now we write down the inhomogeneous Maxwell equation for $\nu = 1$

$$\partial_\mu F^{\mu 1} = -\partial_0 \frac{E_x}{c} + \partial_2 B_z - \partial_3 B_y = -\frac{1}{c^2}\frac{\partial}{\partial t}E_x + \Big(\nabla\times\vec{B}\Big)_x = \mu_0 j_x. \tag{40}$$

**Example: hom. M. eq.** $\mu = 1, \nu = 2, \rho = 3$

$$-\partial_1 B_x - \partial_2 B_y - \partial_3 B_z = -\mathrm{div}\vec{B} = 0 \tag{41}$$

**Action** Finally we mention that the Maxwell equations can be derived form the action $\int \mathcal{L}(A, \partial A) d^4x$, where the integral goes over all of space-time, with the Lagrangian density

$$\mathrm{SI}: \mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu, \qquad \mathrm{cgs}: \mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_\mu j^\mu. \tag{42}$$

# References

[Graf] Lecture Notes Elektrodynamik FS08, Gian Michele Graf, ETH Zurich

[Renner] Lecture Notes Elektrodynamik 2010, Renato Renner, ETH Zurich