General relativity. Problem set 12.

HS 14 *Due:* Tue, December 9, 2014

1. Schwarzschild solution with cosmological term

In class the Schwarzschild metric was found as the static, rotationally invariant solution to the Einstein field equations in vacuum, $R_{\mu\nu} = 0$.

i) Do the same for the case that a cosmological term is included,

$$R_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{1}$$

cf. solution to Problem 8.3. Show that the metric is (de Sitter 1917)

$$ds^2 = \Bigl(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\Bigr)dt^2 - \Bigl(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\Bigr)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\ . \tag{2}$$

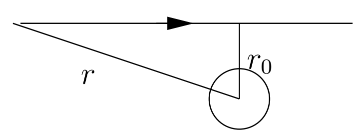
Hint: The ansatz (7.8) is still valid.

ii) Let $m > 0$. For which values of Λ, m is t a time-like coordinate for suitable values of r?

iii) Compute the Newtonian gravitational potential resulting in the weak field limit and compare it with the expression in Problem 8.3 (ii).

2. Time delay in the Schwarzschild metric

Consider a ray passing near the Sun at minimal distance r_0.



Non relativistically it takes light a time $t = \sqrt{r^2 - r_0^2}$, $(c = 1)$ to reach radius r_0 from r (or vice versa).

i) Show that in Schwarzschild coordinates this time is

$$t = \int_{r_0}^{r} \frac{dr}{1 - \frac{2m}{r}} \Bigl(1 - \frac{1 - 2m/r}{1 - 2m/r_0}\Bigl(\frac{r_0}{r}\Bigr)^2\Bigr)^{-1/2}\ . \tag{3}$$

Hint: Use the radial eq. $\dot{r}^2 + V(r) = \mathcal{E}^2$ and express $\dot{r} = dr/d\tau$ by dr/dt using the conservation of $\mathcal{E}$. Establish a relation between $l/\mathcal{E}$ and r_0.

ii) Compute (3) for small m/r_0 and conclude that the time delay $\Delta t = t - \sqrt{r^2 - r_0^2}$ (Shapiro delay, 1964) is

$$\Delta t(r) = 2m \log\Bigl(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\Bigr) + m\Bigl(\frac{r - r_0}{r + r_0}\Bigr)^{1/2} + O(m^2)\ .$$

iii) Let the ray join two planets, e.g. Earth and Venus, at radii r_1 and r_2 on opposite sides of r_0. The round trip delay,

$$\Delta t = 2(\Delta t(r_1) + \Delta t(r_2)) \ ,$$

of a radar signal is measurable. Compute it for $r_1, r_2 \gg r_0$.