General relativity. Problem set 4.

HS 14 *Due:* Tue, October 14, 2014

1. Affine connections

Let an affine connection ∇ on the manifold M be given. Show that $\tilde{\nabla} : (X, Y) \mapsto \tilde{\nabla}_X Y$ is an affine connection too iff the difference $B(X, Y) := \nabla_X Y - \tilde{\nabla}_X Y$ has the property that

$$(\omega, X, Y) \longmapsto \langle \omega, B(X, Y) \rangle \tag{1}$$

is a tensor field of type $\binom{1}{2}$.

We observe a restatement of this fact: Affine connections on M form an affine space over the linear space of tensor fields of that type. (Essentially, a set is an affine space if its elements differ by vectors.)

What does (1) imply for the Christoffel symbols? Show also the application: For any two affine connections $\nabla, \tilde{\nabla}$ the combination $(1 - \alpha)\nabla + \alpha\tilde{\nabla}$ is one, too.

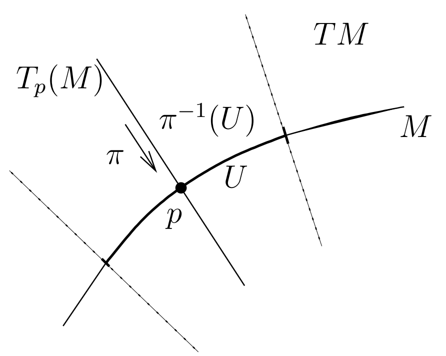
2. A second look at parallel transport

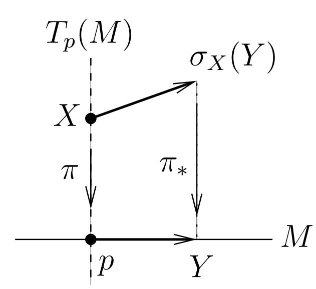
The goal of this exercise is to give a definition of parallel transport not referring to charts. A number of definitions (a-c) alternate with problems (i-iv) which can be solved almost independently.

a) The tangent bundle of a manifold M is the disjoint union of all its tangent spaces, (see left figure)

$$TM = \bigcup_{p \in M} T_p(M) \, .$$

Let π be the projection $\pi : TM \to M$, $X \mapsto \pi(X) = p$ if $X \in T_p(M)$.





The tangent bundle becomes a differentiable manifold of its own by means of charts defined as follows: If $K : U \to \mathbb{R}^n$, $p \mapsto x = (x^1, \ldots, x^n)$ is a chart for the patch $U \subset M$, then a chart for the patch $\pi^{-1}(U) := \bigcup_{p \in U} T_p(M) \subset TM$ is given by

$$\begin{aligned} \tilde{K} : \pi^{-1}(U) &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ X &\longmapsto (x, \underline{X}) \, , \end{aligned}$$

where $\underline{X} = (X^1, \dots X^n)$ are the components of $X \in T_p(M)$ w.r.t. the coordinate basis.

i) Let $\bar{x} \mapsto x$ be a coordinate change on $U \cap \bar{U} \subset M$. Compute the induced coordinate change on $\pi^{-1}(U \cap \bar{U})$. What is the matrix of its partial derivatives?

b) The linear operations on $T_p(M)$ (multiplication by $\lambda \in \mathbb{R}$, addition $+$) induce maps on TM

$$\lambda : TM \to TM, X \mapsto \lambda X \,, \qquad a : TM \boxtimes TM \to TM, (X_1, X_2) \mapsto X_1 + X_2 \,, \tag{2}$$

where $TM \boxtimes TM := \bigcup_{p \in M} T_p(M) \times T_p(M) = \{(X_1, X_2) \in TM \times TM \mid \pi(X_1) = \pi(X_2)\}$.

Consider a curve $\gamma(t) \in M$ having tangent vectors $\dot{\gamma}(t) \in T_{\gamma(t)}$. In class the property that a family of vectors $X(t) \in T_{\gamma(t)}(M)$ is parallel transported along it was formulated by means of a chart K: If $\gamma(t)$ has coordinates $x(t)$ and $X(t)$ has components $\underline{X}(t)$, then

$$\dot{X}^i(t) = -\Gamma^i{}_{lk}(x(t))\dot{x}^l(t)X^k(t) \,. \tag{3}$$

In order to find an intrinsic formulation, note that $\dot{\underline{X}} = (\dot{X}^1, \dots \dot{X}^n)$ are not the components of a vector. Rather, $X(t) \in T_{\gamma(t)}$ is a curve in TM, whence $\dot{X}(t) \in T_{X(t)}(TM)$ has components $(\dot{x}(t), \dot{\underline{X}}(t))$ relative to the chart $\tilde{K}$.

A parallel transport can be viewed as a map

$$\sigma_X : T_{\pi(X)}(M) \longrightarrow T_X(TM) \,, \qquad Y \longmapsto \sigma_X(Y) \,, \; (X \in TM)$$

which is linear in Y and satisfies (see right figure)

$$\pi_* \sigma_X(Y) = Y \,. \tag{4}$$

Moreover it depends on X compatibly with (2): Setting $\tilde{\sigma}(X) = \sigma_X(Y)$ for fixed Y,

$$\lambda_* \tilde{\sigma}(X) = \tilde{\sigma}(\lambda X) \,, \qquad a_*(\tilde{\sigma}(X_1), \tilde{\sigma}(X_2)) = \tilde{\sigma}(a(X_1, X_2)) \,. \tag{5}$$

ii) Write a vector $\mathcal{X} \in T_X(TM)$ in components $(\underline{V}, \underline{W})$ w.r.t. the chart $\tilde{K}$; for short $\mathcal{X} \rightsquigarrow (\underline{V}, \underline{W})$. Show that

$$\sigma_X(Y) \rightsquigarrow \big(\underline{Y}, -\Gamma(x, \underline{Y}, \underline{X})\big)$$

with Γ linear in $\underline{Y}, \underline{X}$, i.e. $\Gamma(x, \underline{Y}, \underline{X})^i = \Gamma^i{}_{lk}(x) Y^l X^k$. The $\Gamma^i{}_{lk}$ may be called Christoffel symbols.

c) Say a vector $X(t)$ is parallel transported along a curve $\gamma(t)$ in M if

$$\pi\big(X(t)\big) = \gamma(t) \,, \qquad \dot{X}(t) = \sigma_{X(t)}(\dot{\gamma}(t)) \,.$$

iii) Show that this is equivalent to (3).

iv) Components of $\mathcal{X} \in T_X(TM)$ transform tensorially by means of the matrix found in (i). Derive from that the non-tensorial transformation law for the Christoffel symbols.