Exercise 1. Goldstone's theorem allows us to think of pions as the pseudo Goldstone bosons of the chiral 2-flavour QCD Lagrangian. The axial current allows to excite a pion out of the vacuum. The matrix element for the one-pion state is given by

$$\langle 0 \mid A_i^\mu(x) \mid \pi_j \rangle = \frac{iF_\pi \delta_{ij} p_\pi^\mu e^{ip_\pi x}}{2(2\pi)^{3/2}\sqrt{2p_\pi^0}}$$

(a) Can you motivate the individual contributions to this formula?

The pion constant F_π could in principle be calculated from QCD using very difficult Lattice simulations. It can also be determined from the interaction of pions with other particles. At low energies we can use an effective Lagrangian to describe the interaction of Standard Model fermions with the symmetry currents $\vec{V}$ and $\vec{A}$. This Lagrangian takes the form

$$\mathcal{L}_{wk} = \frac{iG_{wk}}{\sqrt{2}} \left(V_+^\mu + A_+^\mu\right) \sum_{l=e,\mu,\tau} \bar{l}\gamma_\mu(1+\gamma_5)\nu_l + h.c.$$

Here the sum runs over the three families of leptons and the currents

$$V_\pm^\mu = V_1^\mu \pm iV_2^\mu, \qquad A_\pm^\mu = A_1^\mu \pm iA_2^\mu$$

allow for the exchange of electric charge.

(b) Calculate the decay rate of $\pi \to \mu\nu$ and show that it is given by

$$\Gamma_{\pi\to\mu\nu} = \frac{G_{wk}^2 F_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2}{16\pi m_\pi^3}.$$

(c) The pion decay rate was measured to be $\Gamma_{\pi\to\mu\nu} = (2.6 \times 10^{-8}s)^{-1}$ and the weak coupling was experimentally determined to be $G_{wk} = 1.15\times10^{-5}\mathrm{GeV}^{-2}$. Compute the pion constant F_π. Why is the decay to electrons suppressed?

(d) Make an Ansatz for the matrix-element of the interaction of the axial vector current $\vec{A}_\mu$ with one-nucleon states. Argue that you can constrain the matrix-element using Lorentz invariance, parity- and charge-conservation to

$$\langle p \mid A_+^\mu(x) \mid n \rangle = \frac{e^{iqx}}{(2\pi)^3} \bar{u}_p \left[-i\gamma^\mu\gamma_5 f(q^2) + q^\mu\gamma_5 g(q^2)\right] u_n,$$

where $q^\mu = p_n^\mu - p_p^\mu$.

(e) Neglecting the pion mass the chiral symmetry is exact at the level of the chiral Lagrangian. Use current-conservation and

$$\bar{u}_p(i\not{p}_p + m_N) = (i\not{p}_n + m_N)u_n = 0$$

to derive the identity

$$2m_N f(q^2) = q^2 g(q^2),$$

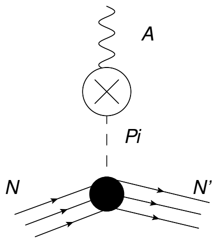
where $m_N = m_p = m_n$.

(f) Meassurements of the decay of neutrons allowed to determine that $f(0) = 1.26$ and thus is non-vanishing for $q \to 0$. What does this imply for the function $g(q^2)$. How can we understand the pole-structure of $g(q^2)$ in terms of a spontaneous breakdown of a global symmetry?

The interaction of a pion and a nucleon can be described via the effective Lagrangian

$$\mathcal{L}_{\pi N} = -2iG_{\pi N}\vec{\pi}\bar{N}\gamma_5\vec{t}N.$$

From this perspective the interaction of an axial current and 2 one-nucleon states can be understood as being pion-mediated. An explicit calculation of the matrix-element yields



$$\langle p \mid A_+^\mu(x) \mid n\rangle \sim \frac{e^{iqx}}{(2\pi)^3}(2iG_{\pi N}\bar{u}_p\gamma_5 u_n)\frac{1}{q^2}q^\mu F_\pi$$

Comparing this result with the contribution to the pole in q^2 from eq. ((d)) we find

$$g(q^2) \sim \frac{G_{\pi N}F_\pi}{q^2}$$

This result allows to derive the so-called Goldberger-Treimann relation

$$G_{\pi N} = \frac{2m_N f(0)}{F_\pi}.$$

Historically, this relation was important to prove the validity of the picture of Goldstone bosons as it represents a link between effective nucleon interactions ($G_{\pi N}$) and fundamental QCD properties (F_π). The pion-nucleon coupling could be measured for example in scattering of $\pi N \to \pi N$ confirming this relation within an accuracy of 10%.