

Integrability in QFT and AdS/CFT

Lecture Notes

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0 Overview

0.1 Introduction

What is integrability?

- ... a peculiar feature of some theoretical physics models.
- ... makes calculations in these models much more feasible in principle and in practice; it is also known as solvability.
- ... allows to compute some quantities exactly and analytically rather than approximately and numerically.
- ... is a hidden enhancement of symmetries which constrain the motion substantially or completely.
- ... is the absence of chaotic motion.
- ... is a colourful mixture of many subjects and techniques from mathematics to physical phenomena.
- ... a lot of fun.

Which classes of models are integrable?

- some classical mechanics models, e.g.: free particle, harmonic oscillator, spinning top, planetary motion,¹
- some (1 + 1)-dimensional classical field theories, e.g.: KdV, sine-Gordon, Einstein gravity, sigma models on coset spaces, classical magnets, string theory.
- some quantum mechanical models, e.g. the quantum versions of the above classical mechanics models.
- some (1 + 1)-dimensional quantum field theories, e.g. most of the quantum counterparts of the above classical field theories, except cases where integrability is spoiled by quantum effects.
- some 2-dimensional models of statistical mechanics, e.g. 6-vertex model, 8-vertex model, alternating sign matrices, loop models, Ising model,
- $D = 4$ self-dual Yang–Mills theory.
- $D = 4$, $N = 4$ maximally supersymmetric Yang–Mills theory in the planar limit and the AdS/CFT dual string theory on $AdS_5 \times S^5$.

One observes that integrability is a phenomenon largely restricted to two-dimensional systems. There are some higher-dimensional exceptions, but most of them have some implicit two-dimensionality (self-duality, planar limit).

¹Most models discussed in lectures and textbooks are in fact integrable, most likely because they can be solved easily and exactly.

0.2 Contents

1. Classical Integrability (1.5h)
2. Integrable Field Theory (2.5h)
3. Integrable Spin Chains (4.5h)
4. Quantum Integrability (3.0h)
5. AdS/CFT Integrability (2.5h)

0.3 Literature

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1 Classical Integrability

Here we discuss integrability for a system of classical mechanics with finitely many degrees of freedom. Although this will not be the main subject of this course, it is very instructive because there is a clear notion of integrability in this case which lays the foundation for the more elaborate cases of field theory and quantum mechanics discussed later.

1.1 Hamiltonian Mechanics

We start by defining a classical mechanics system in Hamiltonian formulation. It consists of a phase space \mathcal{M} of dimension $2n$ and a Hamiltonian function $H : \mathcal{M} \rightarrow \mathbb{R}$. Phase space is defined by a set of coordinates q^k and momenta p_k with $k = 1, \dots, n$.

A solution of the system is a curve $(q^k(t), p_k(t))$ in phase space which obeys the Hamiltonian equations of motion

$$\dot{q}^k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q^k}. \quad (1.1)$$

It is convenient to introduce Poisson brackets which map a pair of functions F, G on phase space to another function on phase space¹

$$\{F, G\} := \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q^k} - \frac{\partial F}{\partial q^k} \frac{\partial G}{\partial p_k}. \quad (1.2)$$

The Poisson brackets are anti-symmetric and they obey the Jacobi identity

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0. \quad (1.3)$$

The Poisson brackets allows to write the equations of motion in a compact and uniform fashion as

$$\frac{d}{dt} q^k = \{H, q^k\}, \quad \frac{d}{dt} p_k = \{H, p_k\}. \quad (1.4)$$

More generally, the time-dependence of a function $F(q, p, t)$ evaluated on a solution $(q^k(t), p_k(t))$ reads²

$$\frac{d}{dt} F = \frac{\partial F}{\partial t} + \{H, F\}. \quad (1.5)$$

¹The Poisson brackets are often defined by specifying the canonical relations $\{p^k, q_l\} = \delta_l^k$ along with the trivial ones $\{p^k, p^l\} = \{q_k, q_l\} = 0$.

²To make proper sense of the above equations of motion one should introduce the coordinate functions $q^k(q, p, t) := q^k$, $p_k(q, p, t) := p_k$ whose partial derivatives w.r.t. time vanish, $\partial q^k / \partial t = \partial p_k / \partial t = 0$.

1.2 Integrals of Motion

For a time-independent Hamiltonian, $\dot{H} = 0$, the function H is an *integral of motion* or *conserved quantity*

$$\frac{d}{dt} H = \frac{\partial H}{\partial t} + \{H, H\} = 0. \quad (1.6)$$

The immediate benefit is that solutions are constrained to a hypersurface of \mathcal{M} defined by $H = E = \text{const.}$ (constant energy). It is therefore easier to find solutions.

Depending on the model, further (time-independent)³ integrals of motion F_k can exist

$$\frac{d}{dt} F_k = \{H, F_k\} \stackrel{!}{=} 0. \quad (1.7)$$

This gives additional constraints $F_k = f_k = \text{const.}$ and motion takes place on an even lower-dimensional hypersurface which is called a *level set*

$$\mathcal{M}_f := \{x \in \mathcal{M}; F_k(x) = f_k\}. \quad (1.8)$$

By construction, the Hamiltonian H is among them and one may identify $F_1 = H$. Additional simplifications come about when the integrals are *in involution* or (*Poisson*) *commute*

$$\{F_k, F_l\} = 0. \quad (1.9)$$

This allows to consistently define solutions (q, p) depending on several time variables t^k such that time-dependence for any function G is determined by⁴

$$\frac{d}{dt^k} G = \frac{\partial G}{\partial t^k} + \{F_k, G\}. \quad (1.10)$$

Finding integrals of motion is all but straight-forward:

- They are often found by trial and error based on a suitable ansatz.
- Noether's theorem implies the existence of a conserved quantity for each global symmetry of the system.⁵

1.3 Liouville Integrability

A system with $2n$ -dimensional phase space \mathcal{M} is called (Liouville) integrable if it has

³Throughout this course we will implicitly assume that functions of phase space have no explicit time dependence.

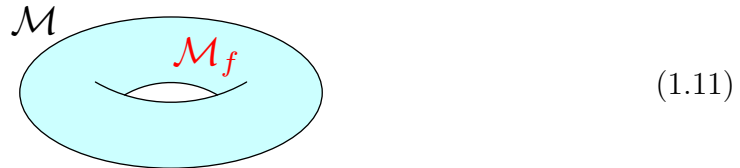
⁴Consistency requires that d/dt^k commutes with d/dt^l which is guaranteed by $\{F_k, F_l\} = 0$ by means of the Jacobi identity.

⁵Additional conserved quantities can often be regarded as a consequence of additional hidden symmetries of the system.

- n independent⁶
- everywhere differentiable
- integrals of motion F_k
- in involution, $\{F_k, F_l\} = 0$.

Such a system is solvable by quadratures, i.e. it suffices to solve a finite number of algebraic equations and integrals.

For integrable systems the following theorem holds: If the level set \mathcal{M}_f is compact, it is diffeomorphic to the n -dimensional torus T^n , the so-called *Liouville torus*.⁷

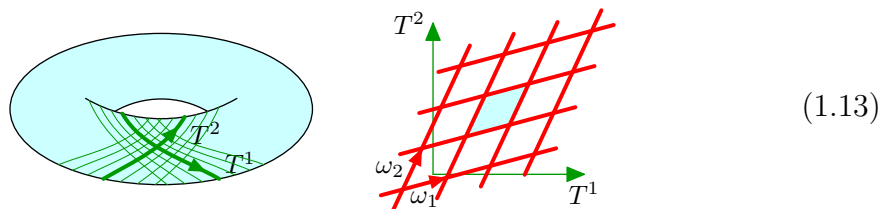


For an integrable system, we can define a set of n time functions T^k on phase space such that $\{F_k, T^l\} = \delta_k^l$. These differential equations define the T^k on each level set. Furthermore, the time functions can be defined across the level sets by imposing the differential equations $\{T^k, T^l\} = 0$. Suitable functions can be constructed thanks to the Jacobi identities. Altogether we have

$$\{F_k, T^l\} = \delta_k^l, \quad \{F_k, F_l\} = \{T^k, T^l\} = 0, \quad (1.12)$$

which tells us that the map $(q^k, p_k) \rightarrow (T^k, F_k)$ is a canonical transformation.

Note that the time functions are in general multiple-valued on phase space. Going around a non-trivial cycle of a level set, the times jump by a definite amount,⁸ given by the period matrix. In that sense, the time functions T^k are uniquely defined on the universal cover \mathbb{R}^n of the level sets. Conversely, the level set is the quotient of \mathbb{R}^n by the lattice defined by the periods.



A useful corollary of integrability is that motion on the level set torus is linear since

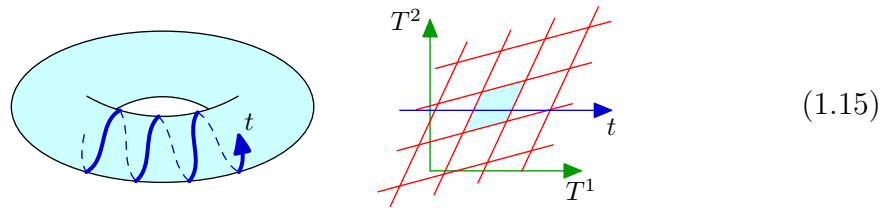
$$\{H, T^k\} = \{F_1, T^k\} = \delta_1^k, \quad (1.14)$$

⁶A function of phase space is called independent of a set of functions if it cannot be written as a function of the values of the other functions. For instance, the total angular momentum $J^2 = J_x^2 + J_y^2 + J_z^2$ is dependent on the components $\{J_x, J_y, J_z\}$ of the angular momentum. Moreover, a constant function is always dependent, even on the empty set of functions.

⁷This theorem follows by considering the vector fields associated to the integral of motions F_k defined by the operation $\{F_k, \cdot\}$. By construction, the vector fields act within the level set and commute with each other. It is well-known that a compact manifold of dimension n which admits n commuting vector fields is diffeomorphic to the torus T^n .

⁸The differential equations determine the time functions locally up to a constant shift.

where we have assumed that H is F_1 . In other words, the solution in the new coordinates is $T^1 = t$ and all other T^k and all F_k are constant.



It would make sense to discuss some basic integrable models in detail, in particular, how to solve them exactly. However, many of them can be found in textbooks of classical mechanics, and it makes no sense to repeat the treatment here.

1.4 Comparison of Classes

Locally we can always define functions (T^k, F_k) such that $\{F_k, T^l\} = \delta_k^l$, $\{F_k, F_l\} = \{T^k, T^l\} = 0$ with the Hamiltonian as $H = F_1$. This means that any system of classical mechanics can be considered integrable in a sufficiently small patch of phase space. In that sense, integrability is a property which depends strongly on the global structure of phase space.

Chaos. However, it may be impossible to continue the integrals of motion consistently to all regions of phase space. In other words, when following the level set along a solution one may end up in the initial region with the level set misaligned with the original one. This is a hallmark feature of *chaotic motion*. An exponential divergence of solutions essentially implies that tracing out nearby solutions would lead to highly complicated hypersurfaces spread out wildly across phase space which could not possibly be described as level sets of globally differentiable functions F_k . Most dynamical systems with more than one degree of freedom, i.e. a phase space of dimension 4 or higher, are chaotic. An example of a chaotic system is the double pendulum.⁹



Integrability. For an integrable system the hypersurfaces match up well globally due to their definition as a level set of differentiable functions F_k . As discussed above, one finds linear motion on the level set torus. Since the torus typically has several periods which are rationally incompatible, the motion of the system is quasi-periodic. All (time-independent) dynamical systems with one degree of freedom are integrable. Further examples include the multi-dimensional HO, the spinning top, planetary motion and classical integrable spin chains.

⁹Its two constituent pendulums will alternate between oscillatory and rotational motion with a seemingly random pattern of repetitions.

Super-Integrability. Some systems have more than n integrals of motion, but evidently only n of them can be in involution. These systems are called *super-integrable*. Here some of the periods of the tori are rationally compatible and therefore the orbits partially close. For a maximally super-integrable system with $2n - 1$ integrals of motion the orbits close and the motion is truly periodic. Dynamical systems with one degree of freedom are in fact maximally super-integrable and therefore have periodic motion. Further examples are Kepler's planetary motion,¹⁰ the spinning top¹¹ and multi-dimensional harmonic oscillators with rationally compatible frequencies.

For non-integrable systems, there may be further useful distinctions that could be made:

- n integrals of motion which are not (all) in involution,
- less than n (but more than one) integrals of motion,
- regions of phase space of a chaotic system which admit integrability.

In this lecture series we will only be interested in the fully integrable cases. Super-integrability may occur accidentally, but we will not pay attention to it.

1.5 Structures of Integrability

In the following we shall introduce two important structures for integrable models that will (later in the context of integrable field theories) help us investigate and solve the model.

Lax Pair. Integrable systems are often formulated in terms of a *Lax pair*. A Lax pair is a pair of square matrices L, M whose entries are functions of phase space. The characteristic property of Lax pairs is that the equation

$$\frac{d}{dt} L = [M, L] \tag{1.17}$$

is equivalent to the complete set of equations of motion.

If a Lax pair exists for a classical mechanics system, the matrix L can be used to generate a tower of integrals of motion F_k

$$F_k = \text{tr } L^k. \tag{1.18}$$

These quantities are trivially conserved due to the cyclicity of the trace

$$\frac{d}{dt} F_k = k \text{tr } L^{k-1} [M, L] = 0. \tag{1.19}$$

¹⁰In addition to the angular momentum there is the Runge-Lenz vector which is orthogonal to the angular momentum vector. This amounts to 5 integrals of motion and maximal super-integrability.

¹¹The spinning top has 3 integrals of motion in involution H, J^2 and J_z and is therefore integrable. Among the two further components of the angular momentum vector only one is independent of J^2 and J_z .

For systems with finitely many degrees of freedom, only finitely many of the generated charges can be independent.

Alternatively, the Lax equation is equivalent to the statement that time evolution of L is generated by a similarity transformation. Therefore the eigenvalue spectrum and the characteristic polynomial of L are conserved. Note that the latter is a function of the F_k .

Having a Lax pair formulation of integrability is very convenient, but

- inspiration is needed to find it,
- its structure is hardly transparent,
- it is not at all unique,
- the size of the matrices is not immediately related to the dimensionality of the system.

Therefore, the concept of Lax pairs does not provide a means to decide whether any given system is integrable (unless one is lucky to find a sufficiently large Lax pair).

Classical r-matrix. For integrability we not only need sufficiently many global integrals of motion F_k , but they must also be in involution, $\{F_k, F_l\} = 0$. In the formulation of integrability in terms of a Lax pair $L, M \in \text{End}(V)$, this is equivalent to the statement

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \quad (1.20)$$

The statement is defined on the tensor product space $\text{End}(V \otimes V)$ of two matrices, and the *classical r-matrix* r_{12} is a particular element of this space whose entries are functions on phase space.¹² Furthermore, $L_1 := L \otimes 1$, $L_2 := 1 \otimes L$, and $r_{21} := P(r_{12})$ denotes the permutation of the two spaces for the r-matrix. Note that the r-matrix is by no means uniquely defined by the equation.¹³ Much like for the Lax pair, there is no universal method to obtain the r-matrix.

From the above equation it follows straight-forwardly that

$$\{\text{tr } L^k, \text{tr } L^l\} = 0. \quad (1.21)$$

There is a useful graphical representation of the equation where matrices are objects with one ingoing and one outgoing leg. Connecting two legs corresponds to a product of matrices, whereas two matrices side by side correspond to a tensor product. Consequently, the classical r -matrix will be an object with two ingoing

¹²The tensor product $A \otimes B$ of two matrices with elements A^a_c, B^b_d has the elements $(A \otimes B)^{ab}_{cd} = A^a_c B^b_d$, where ab and cd are combined indices enumerating a basis for the tensor product space.

¹³For example, one can add an operator of the form $1 \otimes X + L \otimes Y$ where X and Y are arbitrary matrices.

and outgoing legs, and the above equation reads

$$\begin{aligned}
\left\{ 1 \text{---} L \text{---} 1, 2 \text{---} L \text{---} 2 \right\} = & \begin{array}{c} \begin{array}{c} 1 \\ 2 \end{array} \text{---} \begin{array}{c} L \\ r \end{array} \text{---} \begin{array}{c} 1 \\ 2 \end{array} \\ \text{---} \begin{array}{c} L \\ r \end{array} \text{---} \begin{array}{c} 1 \\ 2 \end{array} \end{array} - \begin{array}{c} \begin{array}{c} 1 \\ 2 \end{array} \text{---} \begin{array}{c} L \\ r \end{array} \text{---} \begin{array}{c} 1 \\ 2 \end{array} \\ \text{---} \begin{array}{c} L \\ r \end{array} \text{---} \begin{array}{c} 1 \\ 2 \end{array} \end{array} \\
- & \begin{array}{c} \begin{array}{c} 2 \\ 1 \end{array} \text{---} \begin{array}{c} L \\ r \end{array} \text{---} \begin{array}{c} 2 \\ 1 \end{array} \\ \text{---} \begin{array}{c} L \\ r \end{array} \text{---} \begin{array}{c} 2 \\ 1 \end{array} \end{array} + \begin{array}{c} \begin{array}{c} 2 \\ 1 \end{array} \text{---} \begin{array}{c} L \\ r \end{array} \text{---} \begin{array}{c} 2 \\ 1 \end{array} \\ \text{---} \begin{array}{c} L \\ r \end{array} \text{---} \begin{array}{c} 2 \\ 1 \end{array} \end{array} . \tag{1.22}
\end{aligned}$$

Many relationship can be conveniently expressed and proved using this graphical notation. We shall make extensive use of it in the context of integrable spin chains.

Example. Consider a harmonic oscillator with frequency ω . A Lax pair is given by

$$L = \begin{pmatrix} +p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -\frac{1}{2}\omega \\ +\frac{1}{2}\omega & 0 \end{pmatrix}. \tag{1.23}$$

The Lax equation is equivalent to the equation of motion of the harmonic oscillator

$$\dot{p} = -\omega^2 q, \quad \omega \dot{q} = \omega p. \tag{1.24}$$

The resulting integrals of motion read

$$\begin{aligned}
F_1 &= 0, \\
F_2 &= 2p^2 + 2\omega^2 q^2 = 4H, \\
F_3 &= 0, \\
F_4 &= 2(p^2 + \omega^2 q^2)^2 = 8H^2, \\
&\dots \tag{1.25}
\end{aligned}$$

Here F_1 and F_3 are trivial and can be ignored. The first and only non-trivial integral of motion F_2 is the Hamiltonian. The higher even powers are merely powers of the Hamiltonian which are not independent integrals of motion.

For this system, a classical r-matrix is given by

$$r_{12} = \frac{1}{q} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \frac{1}{q} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.26}$$

The above commutators with the Lax matrix L then agrees precisely with the Poisson brackets

$$\{L_1, L_2\} = \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.27}$$

2 Integrable Field Theory

2.1 Classical Field Theory

Next we consider classical mechanics of a one-dimensional field $\varphi(x)$. Together with time-evolution, this amounts to a problem of $(1+1)$ -dimensional fields $\varphi(t, x)$. The phase space for such models is infinite-dimensional,¹ thus integrability requires infinitely many integrals of motion in involution. Comparing infinities is subtle, so defining integrability requires care. Since there is no clear notion of integrability for field theories, we will be satisfied with the availability of efficient constructive methods for solutions. Whether or not a model is formally integrable will be of little concern.

Most random field theory models are clearly non-integrable, but there are several well-known models that are integrable:

- Korteweg–de Vries (KdV) equation

$$\dot{u} = 6uu' - u'''. \quad (2.1)$$

This is perhaps the prototype integrable field theory. It models surface waves in shallow water.

- Non-linear Schrödinger equation

$$i\dot{\psi} = -\psi'' + 2\kappa|\psi|^2\psi. \quad (2.2)$$

- sine-Gordon equation (relativistic)

$$\ddot{\phi} - \phi'' + \frac{m^2}{\beta} \sin(\beta\phi) = 0. \quad (2.3)$$

This equation has many generalisations: non-linear sigma models on coset spaces.

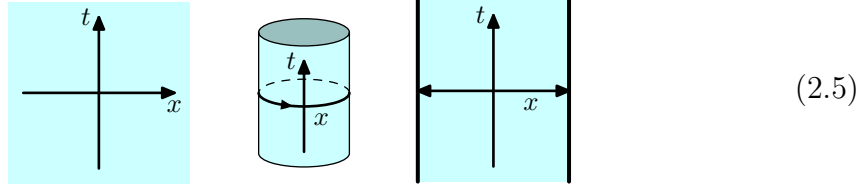
- classical Heisenberg magnet (Landau–Lifshitz equation)

$$\dot{\vec{S}} = -\kappa\vec{S} \times \vec{S}'', \quad \vec{S}^2 = 1. \quad (2.4)$$

In addition to the bulk equations of motion, a complete definition of the model also requires the specification of *boundary conditions*. The most common choices are:

- infinite spatial extent with rapidly decaying fields (or derivatives) as $x \rightarrow \infty$,
- closed or periodic boundary conditions with $x \equiv x + L$,
- open boundaries with Dirichlet or Neumann conditions $\phi = \text{const.}$ or $\phi' = 0$.

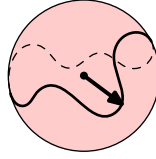
¹Note that a time slice is a field which can be Taylor or Fourier expanded leading to infinitely many independent coefficients.



(2.5)

Boundary conditions may also be twisted in some way or combined differently.

Heisenberg Magnet. The classical Heisenberg magnet is a model of a one-dimensional magnetic material with a magnetisation or spin vector field $\vec{S}(t, x)$. The spin vector has a constant length, $\vec{S}^2 = 1$.²



(2.6)

The energy depends on the alignment of nearby spins, the simplest ansatz is³

$$H = \frac{\kappa}{2} \int dx \vec{S}'^2. \quad (2.7)$$

A suitable Poisson structure is⁴

$$\{S^a(x), S^b(y)\} = \varepsilon^{abc} S^c(x) \delta(x - y). \quad (2.8)$$

The equations of motion are the so-called Landau–Lifshitz equations

$$\dot{\vec{S}}(x) = \{H, \vec{S}(x)\} = -\kappa \vec{S}(x) \times \vec{S}''(x). \quad (2.9)$$

The system is formulated without making reference to a preferred vector. Therefore it has a global rotational symmetry $\vec{S}(x) \mapsto R\vec{S}(x)$ with $R \in \text{SO}(3)$. This leads to a Noether current \vec{J}_α and associated Noether charge \vec{Q}

$$\vec{J}_t = \vec{S}, \quad \vec{J}_x = -\kappa \vec{S} \times \vec{S}', \quad \vec{Q} = \int dx \vec{S}. \quad (2.10)$$

The current and charge are conserved $\vec{J}'_x - \dot{\vec{J}}_t = 0$ and $\dot{\vec{Q}} = 0$ provided the field \vec{S} satisfies the equations of motion.

This model is integrable, therefore there are many additional integrals of motion.

²The field $\vec{S}(t, x)$ can also be viewed as the evolution of a one-dimensional curve on a two-dimensional sphere S^2 .

³Note that $\vec{S} \cdot \vec{S}' = 0$ due to $\vec{S}^2 = 1$.

⁴The above Hamiltonian along with the Poisson structure follows from a Lagrange function which is somewhat subtle.

Lax Connection. We want to set up a Lax pair to describe the integrals of motion for the field theory. In field theory we require infinitely many conserved quantities so the Lax pair has to be infinite-dimensional. Alternatively, we can set up a Lax pair with an additional continuous parameter λ , the so-called *spectral parameter*. Taylor expansion in λ then leads to an infinite tower of conserved quantities.

Furthermore, the we prefer to formulate in terms of local objects in a field theory. Therefore introduce the *Lax connection* $A_\alpha(\lambda; t, x)$

$$\begin{aligned} A_x(\lambda) &= -\frac{i}{\lambda} \vec{\sigma} \cdot \vec{S}, \\ A_t(\lambda) &= \frac{i\kappa}{\lambda} \vec{\sigma} \cdot (\vec{S} \times \vec{S}') + \frac{2i\kappa}{\lambda^2} \vec{\sigma} \cdot \vec{S}. \end{aligned} \quad (2.11)$$

In our case, the Lax connection is a 2×2 matrix-valued field whose entries depend on λ and are functions of phase space. Here $\vec{\sigma}$ is the triplet of 2×2 traceless hermitian Pauli matrices. The Lax connection satisfies the flatness condition for all λ

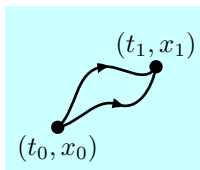
$$\dot{A}_x(\lambda) - A'_t(\lambda) + [A_x(\lambda), A_t(\lambda)] = 0 \quad (2.12)$$

provided that the equations of motion hold and $\vec{S}^2 = 1$.

As always, the Lax connection is not unique. However, a useful recipe to construct it, is to make an ansatz in terms of the components of a Noether current, J_x and J_t in our case, and constrain the coefficients by means of the flatness condition.

It is convenient to work with the Lax connection using the language of differential forms. It is a $\mathfrak{su}(2)$ connection one-form $A(\lambda) = A_x(\lambda)dx + A_t(\lambda)dt$ which obeys $dA(\lambda) = A(\lambda) \wedge A(\lambda)$.

Lax Monodromy. We wish to obtain integrals of motion via the Lax pair. The latter can be constructed from the parallel transport operator (path-ordered integral, Wilson line) of the Lax connection $A(\lambda)$

$$U(\lambda; t_1, x_1; t_0, x_0) := \text{P exp} \int_{(t_0, x_0)}^{(t_1, x_1)} A(\lambda). \quad (2.13)$$



Due to flatness it is invariant under continuous deformations of the path contour between (t_0, x_0) and (t_1, x_1) . Moreover, shifting the end points amounts to simple differential equations⁵

$$\partial_\alpha^1 U^{10} = A^1 U^{10}, \quad \partial_\alpha^0 U^{10} = -U^{10} A^0. \quad (2.14)$$

Here the upper indices 0 and 1 represent the points (t_0, x_0) and (t_1, x_1) , respectively.

⁵In fact, these equations can be viewed as the defining properties of U^{10} .

The Lax pair is constructed from the parallel transport operator, but we have to take the boundary conditions into account. The simplest choice are periodic boundaries, $\vec{S}(x + L) = \vec{S}(x)$. In this case the Lax pair is defined as

$$L(\lambda) = \text{P exp} \int_0^L dx A_x(\lambda), \quad M(\lambda) = A_t(\lambda)|_{x=0}. \quad (2.15)$$


The matrix $L(\lambda)$ is also known as the *monodromy matrix*. The above differential equations for U^{10} imply the Lax equation⁶

$$\dot{L}(\lambda) = [M(\lambda), L(\lambda)]. \quad (2.16)$$

Integrals of Motion. The eigenvalues or equivalently the traces of powers of L are conserved

$$F_k(\lambda) = \text{tr} L(\lambda)^k. \quad (2.17)$$


One can expand them around some point λ_0 , e.g. $\lambda_0 = \infty$, to obtain an infinite tower of conserved quantities

$$F_k(\lambda) = \sum_{r=0}^{\infty} \frac{F_k^{(r)}}{\lambda^r}. \quad (2.18)$$

For completeness, we need to show that they are in involution. This follows from a slightly different relationship for Lax matrices with spectral parameters

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda) \otimes L_2(\mu)]. \quad (2.19)$$

with the parameter-dependent classical r-matrix

$$r_{12}(\lambda, \mu) = \frac{\sigma^k \otimes \sigma^k}{2(\lambda - \mu)}. \quad (2.20)$$

The latter satisfies the classical Yang–Baxter equation⁷

$$\begin{aligned} & [r_{12}(\lambda - \mu), r_{13}(\lambda - \rho)] \\ & + [r_{12}(\lambda - \mu), r_{23}(\mu - \rho)] \\ & + [r_{13}(\lambda - \rho), r_{23}(\mu - \rho)] = 0. \end{aligned} \quad (2.21)$$

The above relations imply that

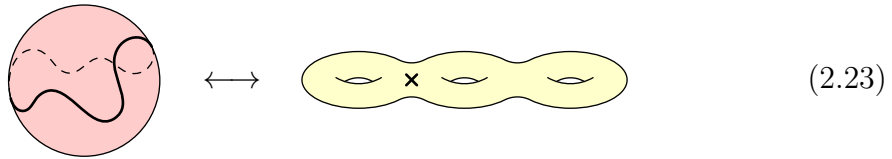
$$\{F_k(\lambda), F_l(\mu)\} = \{F_k^{(r)}, F_l^{(s)}\} = 0. \quad (2.22)$$

⁶Here we set $t_0 = t_1 = t$ and $x_0 = 0 \equiv x_1 = L$ so that the time derivative acts on both t_0 and t_1 and $A_t(\lambda)$ is the same at x_0 and x_1 due to periodicity.

⁷This is the simplest form of classical Yang–Baxter equation. There are various modifications for different types of integrable models.

2.2 Spectral Curves

For any solution $\vec{S}(t, x)$ of the equations of motion we know how to compute the monodromy matrix $L(\lambda)$. It contains a lot, perhaps all, information on the integrals of motion. Let us therefore investigate $L(\lambda)$. In particular, the dependence on the spectral parameter $\lambda \in \mathbb{C}$ reveals many properties of the solution by means of the so-called spectral curve. This information is very useful because it allows to construct suitable spectral curves from scratch and thus learn about the conserved charges of a solution without constructing the latter. One can even reconstruct the solution from the spectral curve with some additional dynamical data.



Riemann Sheets. The eigenvalues $\omega_{1,2}(\lambda)$ of $L(\lambda)$ constitute integrals of motion. They are related to the traces of powers of $L(\lambda)$ as follows

$$F_1(\lambda) = \omega_1(\lambda) + \omega_2(\lambda), \quad F_2(\lambda) = \omega_1(\lambda)^2 + \omega_2(\lambda)^2. \quad (2.24)$$

The inverse relationship reads

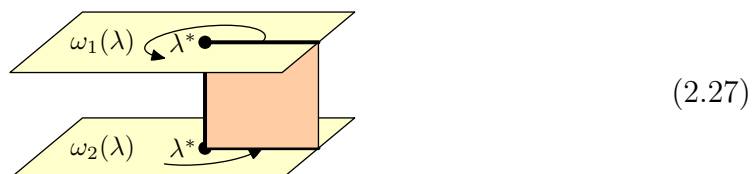
$$\omega_{1,2}(\lambda) = \frac{1}{2}F_1(\lambda) \pm \sqrt{\frac{1}{2}F_2(\lambda) - \frac{1}{4}F_1(\lambda)^2}. \quad (2.25)$$

By construction, the functions $F_k(\lambda)$ are analytic (holomorphic) in λ , except for the special point $\lambda = 0$. The same therefore holds for the eigenvalue functions $\omega_{1,2}(\lambda)$ almost everywhere except for square root branch points λ_k^* of the above relations. At these points the two eigenvalues coincide,

$$\omega_1(\lambda_k^*) = \omega_2(\lambda_k^*). \quad (2.26)$$

Square root singularities are special points for a complex function because they have a full angle of rotation of 4π . A rotation of 2π around these points interchanges ω_1 and ω_2 . There is nothing wrong with such a behaviour because the two eigenvalues are equivalent by all means. A rotation by 2π merely changes our labelling of the eigenvalues which is inconsequential.

The eigenvalue function $\omega(\lambda)$ can thus be viewed as a function on a two-fold cover of the complex plane, with ω_1 and ω_2 being the two Riemann sheets of the function. Importantly, the sheets are joined along branch cuts which connect the branch points λ_k^* in pairs.



The number and locations λ_k^* of the branch points depends on the underlying solution $\vec{S}(t, x)$ in a very non-trivial fashion. Conversely, the locations of the branch cuts determine the functions $\omega(\lambda)$ as we shall show later. Therefore they determine many (if not all) of the integrals of motion and classify solutions $\vec{S}(t, x)$.

Let us determine further properties of the function $\omega(\lambda)$.

Singularity at $\lambda = 0$. Recall that the monodromy matrix $L(\lambda)$ was constructed by means of the matrix

$$A_x(\lambda) = -\frac{i}{\lambda} \vec{\sigma} \cdot \vec{S}. \quad (2.28)$$

It has a pole at $\lambda = 0$ which leads to an essential singularity in $L(\lambda)$. We would like to understand the nature of this singularity better.

To that end, we should diagonalise the connection $A_x(\lambda, x)$ at all x by means of a suitable rotation $U(\lambda, x)$

$$\partial_x + A'_x(\lambda, x) = U(\lambda, x)^{-1} (\partial_x + A_x(\lambda, x)) U(\lambda, x). \quad (2.29)$$

The rotation matrix is determined such that $A'_x \sim \sigma^3$ is diagonal. This rotation matrix $U(\lambda, x)$ can be perfectly regular at $\lambda = 0$. To leading order in λ a diagonal A'_x is achieved by a rotation which satisfies

$$U(0, x)^{-1} (\vec{\sigma} \cdot \vec{S}(x)) U(0, x) = \sigma^3 |\vec{S}(x)| = \sigma^3. \quad (2.30)$$

The resulting connection reads

$$\partial_x + A'_x(\lambda, x) = -\frac{i}{\lambda} \sigma^3 + \mathcal{O}(\lambda^0) \quad (2.31)$$

and the transformed monodromy matrix is now computed as a plain integral without path ordering

$$L'(\lambda) = \exp \int_0^L dx A'_x(\lambda, x). \quad (2.32)$$

Since the monodromy matrices are related by a plain similarity transformation

$$L(\lambda) = U(\lambda, L)^{-1} L'(\lambda) U(\lambda, 0) = U(\lambda, 0)^{-1} L'(\lambda) U(\lambda, 0), \quad (2.33)$$

we can now easily read off the singular behaviour of the eigenvalues at $\lambda = 0$

$$\omega_{1,2}(\lambda) = \exp \left(\pm \frac{iL}{\lambda} + \mathcal{O}(\lambda^0) \right). \quad (2.34)$$

The higher orders at $\lambda = 0$ can be obtained by a careful analysis involving a λ -dependent rotation $U(\lambda, x)$. This is somewhat laborious, and we shall skip the analysis. Importantly, the resulting conserved charges are *local integrals of motion*. The property of locality is closely related to the pole singularity in $A_x(\lambda)$. In our case the lowest few charges are:

- the total momentum P at $\mathcal{O}(\lambda^0)$,
- the total energy E at $\mathcal{O}(\lambda^1)$,
- higher local charges Q_k involving more than two spatial derivatives at $\mathcal{O}(\lambda^{k-1})$.

Quasi-Momentum and Spectral Curve. For a later reconstruction of the function $\omega(\lambda)$ the existence of essential singularities is inconvenient. They can be removed by considering the logarithm of the function $\omega(\lambda)$ which is known as the *quasi-momentum* $q(\lambda)$

$$q(\lambda) := -i \log \omega(\lambda). \quad (2.35)$$

Evidently, the quasi-momentum has single poles at $\lambda = 0$ with residue $\pm L$.

Note that the quasi-momentum $q(\lambda)$ has inherited the ambiguity of the complex logarithm, and is therefore defined only modulo shifts of 2π . Evidently, one will choose the function to be analytic almost everywhere, but in addition to switching sheets at the existing branch cuts of $\omega(\lambda)$, it can jump by multiples of 2π

$$q_1 \leftrightarrow q_2 + 2\pi n. \quad (2.36)$$

The characteristic number n is constant along the branch cut.

To get rid of these ambiguities, it makes sense to consider the derivative of the quasi-momentum q' or dq as a differential form,

$$q'(\lambda) = -i\omega'(\lambda)/\omega(\lambda). \quad (2.37)$$

This function has only two sheets and algebraic type singularities. It can be viewed as a complex curve, the so-called *spectral curve*. It is therefore ideally suited for complex analysis and for construction purposes.

Note that the curve has inherited some properties from its construction via $\omega(\lambda)$. Let us list them:

- All closed periods of $dq(\lambda)$ on the Riemann surface must be multiples of 2π due to its definition as a logarithmic derivative

$$\oint dq = \oint d\lambda q'(\lambda) \in 2\pi\mathbb{Z}. \quad (2.38)$$

- Any point-like singularities cannot have a residue, i.e. they must be poles of higher degree. A pole with a residue requires $q(\lambda)$ to have a logarithmic singularity and thus $\omega(\lambda)$ to have a pole or a zero. This is in conflict with the group nature of the monodromy $L(\lambda)$.
- There is a double pole at $\lambda = 0$ without a residue for the single pole

$$q'_{1,2}(\lambda) = \mp \frac{L}{\lambda^2} + \frac{0}{\lambda} + \dots \quad (2.39)$$

- The function $q'(\lambda)$ has branch cuts which end in inverse square root branch points.

Special Properties. The matrix $L(\lambda)$ has a further special property which follows from a property of $A_x(\lambda)$ and which influences the behaviour of $\omega(\lambda)$ and $q(\lambda)$.

We know that $A_x \sim \vec{\sigma} \cdot \vec{S}$ in a traceless matrix. After integration and exponentiation we derive

$$\det L(\lambda) = 1, \quad \omega_1(\lambda) \omega_2(\lambda) = 1, \quad F_2 = F_1^2 - 2. \quad (2.40)$$

For the quasi-momentum it implies that the two sheets differ merely by their sign and potentially by a shift of a multiple of 2π . In order to fix the shift ambiguity on one sheet, we can define the second sheet to be the negative of the first sheet without a shift

$$q_2 = -q_1. \quad (2.41)$$

Passing through a branch cut therefore must include a potential shift by 2π

$$q \leftrightarrow -q + 2\pi n. \quad (2.42)$$

Expansion at $\lambda = \infty$. Another distinguished point is $\lambda = \infty$ where $A_x(\lambda)$ vanishes. The expansion of $L(\lambda)$ is therefore straight-forwardly the expansion of the exponential

$$L(\lambda) = \exp\left(-\frac{i}{\lambda} s\vec{\sigma} \cdot \vec{Q} + \mathcal{O}(1/\lambda^2)\right), \quad (2.43)$$

where \vec{Q} is the Noether charge for rotations

$$\vec{Q} = \int_0^L dx \vec{J}_t, \quad \vec{J}_t = \vec{S}. \quad (2.44)$$

For the quasi-momentum it implies

$$q(\lambda) = \pm \frac{1}{\lambda} |\vec{Q}|. \quad (2.45)$$

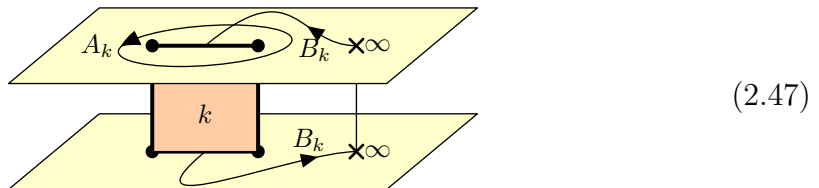
Here we have used and fixed the freedom to shift by multiples of 2π by setting $q(\infty) = 0$.

As an aside, the higher powers of $1/\lambda$ in $L(\lambda)$ correspond to multi-local conserved charges such as

$$\int_0^L dx \int_0^x dx' \vec{S}(x) \times \vec{S}(x'). \quad (2.46)$$

Periods and Moduli. The locations λ_k^* of the branch points determine the spectral curve, but they are not immediately telling much about the physical properties of the underlying solution. There are other quantities which are much more suitable: *periods*.

We know that the periods of dq are integer multiples of 2π . To be more concrete, we choose a convenient basis of cycles on the Riemann surface: There is a cycle around each branch cut, these are called the *A-cycles* A_k . Furthermore there is a cycle through each cut, these are called the *B-cycles* B_k .



In this assignment, the distinguished point at $\lambda = \infty$ can be viewed as an infinitesimally short cut. It serves as the distinguished cut which does not contribute to the counting of cycles because

- the combination of all A-cycles combines to an inverse cycle around the remaining singular point $\lambda = \infty$ and
- all B-cycles close though the “cut” at $\lambda = \infty$.

All A-periods vanish while the B-periods yield integers⁸

$$\oint_{A_k} dq = 0, \quad \int_{B_k} dq = 2\pi n_k. \quad (2.48)$$

The integers n_k describe the jump of the quasi-momentum $q(\lambda)$ at the branch cuts. They are called *mode numbers*.

We can measure another characteristic number for each branch cut as the A-period of λdq , the so-called *filling* K_k

$$K_k = \frac{1}{2\pi i} \oint_{A_k} \lambda dq. \quad (2.49)$$

It is a measure of the length of the branch cut, and unlike n_k it takes continuous values. Note that quantisation of the classical theory renders these numbers to be quantised as integers, too.

Finite Gap Construction. Let us summarise the properties of the spectral curve $q'(\lambda)$:

- The function has two Riemann sheets, it is single-valued on the Riemann surface, the sum of the Riemann sheets is zero.
- The function has branch points λ_k^* of the type $1/\sqrt{\lambda - \lambda_k^*}$.
- There is a fixed pole $\pm L/\lambda^2 + 0/\lambda$ at $\lambda = 0$.
- The asymptotic behaviour at $\lambda \rightarrow 0$ is $\sim 1/\lambda^2$.
-

For spectral curves with finitely many cuts (“finite gap”) we can make a general ansatz as an algebraic curve

$$q'(\lambda) = \pm \frac{P_N(\lambda)}{\lambda^2 \sqrt{Q_N(\lambda)}}, \quad (2.50)$$

where P_N and Q_{2N} are polynomials of degree N and $2N$, respectively, with $2N + 2$ free parameters in total. This ansatz automatically satisfies several of the above properties, the remaining properties constrain some of the parameters as follows:

- N A-periods $\oint dq = 0$,
- N B-periods $\int dq = 2\pi n_k$,
- N fillings $\oint \lambda dq \sim K_k$,

⁸For integer periods one can always make at least half of them vanish by a suitable choice of independent cycles and thus of Riemann sheets and cuts.

- 1 coefficient of the $1/\lambda^2$ pole at $\lambda = 0$,
- 1 ambiguity of overall rescaling of P and \sqrt{Q} .

We learn that all degrees of freedom are fixed by the knowledge of the (discrete) mode numbers n_k and the (continuous) fillings K_k . All integrals of motion (momentum, energy, spin, higher charges) follow from this finite gap solution.

This classifies solutions with finite genus N . One could view the more general spectral curves with infinitely many cuts as the limiting case $N \rightarrow \infty$.

Physics of Spectral Curves. Finite genus spectral curves are specified by one discrete mode number n_k and one continuous filling K_k for each cut. This matches qualitatively with the spectrum for $(1+1)$ -dimensional field theories with closed boundary conditions, such as string theory. Let us discuss the latter:

For solutions \vec{S} near a trivial vacuum solution \vec{S}_0 one could make an ansatz in terms of Fourier modes

$$\vec{S} = \vec{S}_0 + \sum_n \vec{\alpha}_n \exp(2\pi i n x / L). \quad (2.51)$$

Here, the mode numbers n are discrete whereas the amplitudes $\vec{\alpha}_n$ are continuous.



$$(2.52)$$

Finite gap solutions represent solutions where only a finite number of Fourier modes n_k are active, $K_k \sim |\alpha_k|^2 > 0$. Note that for a non-linear problem⁹ the Fourier mode expansion leads to complicated non-linear relationships of the α 's. The spectral curve automatically takes care of this complication. It can be viewed as a non-linear version of the Fourier transformation by means of complex analysis which is perfectly adapted to our physics model.

⁹In our model, the constraint $\vec{S}^2 = 1$ is responsible for non-linearity.

3 Integrable Spin Chains

We now proceed to integrable quantum mechanical models. They are instructive because:

- they form a large class of integrable models,
- they can be treated uniformly,
- they have many parameters to tune,
- short chains are genuine quantum mechanical models,
- long chains approximate (1 + 1)D quantum field theories,
- for large quantum numbers they are approximated by classical models,
- they model magnetic materials.

Magnets. Ansatz: A magnetic material consists of many microscopic magnets, e.g. atoms with spin. The energy of the material depends on the configuration of nearby spins.

nearby spins	ferromagnet	anti-ferromagnet	(3.1)
opposite alignment $\uparrow\downarrow$	high energy	low energy	
equal alignment $\uparrow\uparrow$	low energy	high energy	

Two well known models of magnets are:

- Ising model, a model of statistical mechanics. It consists of a lattice of spins taking values \uparrow, \downarrow . The alignment of nearest neighbours determines the energy.
- Heisenberg chain, a quantum mechanical model. It consists of a chain of spin states $|\uparrow\rangle, |\downarrow\rangle$. The Hamiltonian acts on nearest neighbours.

In the following we shall discuss the Heisenberg spin chain in detail.

3.1 Heisenberg Spin Chain

Let us start by introducing the model and investigating its spectrum.

Setup. A single spin state can be $|\downarrow\rangle$ or $|\uparrow\rangle$ or any complex linear combination of these two. In other words, a spin is described by an element of the vector space

$$\mathbb{V} = \mathbb{C}^2. \tag{3.2}$$

A spin chain of length L is the L -fold tensor product

$$\mathbb{V}^{\otimes L} = \mathbb{V}_1 \otimes \dots \otimes \mathbb{V}_L. \tag{3.3}$$

This space serves as the Hilbert space of our model. It has finite dimension 2^L . A basis is given by the “pure” states, e.g.

$$|\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\rangle. \tag{3.4}$$

The Hamiltonian operator $H : \mathbb{V}^{\otimes L} \rightarrow \mathbb{V}^{\otimes L}$ is homogeneous and acts on nearest neighbours

$$\mathcal{H} = \sum_k \mathcal{H}_{k,k+1}, \quad \mathcal{H}_{k,l} : \mathbb{V}_k \otimes \mathbb{V}_l \rightarrow \mathbb{V}_k \otimes \mathbb{V}_l. \quad (3.5)$$

The pairwise kernel $\mathcal{H}_{k,l}$ for the Heisenberg chain reads

$$\mathcal{H}_{k,l} = \lambda_0(1 \otimes 1) + \lambda_x(\sigma^x \otimes \sigma^x) + \lambda_y(\sigma^y \otimes \sigma^y) + \lambda_z(\sigma^z \otimes \sigma^z). \quad (3.6)$$

It is integrable for all values of the coupling constants $\lambda_0, \lambda_x, \lambda_y, \lambda_z$. Several useful cases can be distinguished:

- The most general (and most complicated) case is $\lambda_x \neq \lambda_y \neq \lambda_z \neq \lambda_x$: This is the so-called “XYZ” model.
- Many simplifications occur for $\lambda_x = \lambda_y \neq \lambda_z$: This is the so-called “XXZ” model.
- Symmetry is enhanced for $\lambda_x = \lambda_y = \lambda_z$: This is the so-called “XXX” model.

We shall mainly use the XXX model with the choice¹

$$\lambda_0 = -\lambda_x = -\lambda_y = -\lambda_z = \frac{1}{2}\lambda. \quad (3.7)$$

With this choice the Hamiltonian kernel reads

$$\mathcal{H}_{k,l} = \lambda(\mathcal{I}_{k,l} - \mathcal{P}_{k,l}), \quad (3.8)$$

where $\mathcal{I}_{k,l}$ is the identity operator and $\mathcal{P}_{k,l}$ the permutation on the two equivalent spaces \mathbb{V}_k and \mathbb{V}_l . Note that $\lambda > 0$ implies ferromagnetic behaviour whereas $\lambda < 0$ implies anti-ferromagnetic behaviour.²

Boundary Conditions. To complete the definition of the model, we must specify the boundary conditions. Typical choices are

- open chain:

$$\mathcal{H} = \sum_{k=1}^{L-1} \mathcal{H}_{k,k+1}, \quad (3.9)$$

- closed chain: identify sites periodically such that $\mathbb{V}_{L+1} = \mathbb{V}_1$

$$\mathcal{H} = \sum_{k=1}^L \mathcal{H}_{k,k+1}, \quad (3.10)$$

- infinite chain:

$$\mathcal{H} = \sum_{k=-\infty}^{+\infty} \mathcal{H}_{k,k+1}. \quad (3.11)$$

¹The value of λ_0 is largely irrelevant because it merely induces an overall shift of all energies. Our choice sets the energy of a reference state to zero.

²We shall be interested in all states of the model, hence the difference between the ferromagnetic and anti-ferromagnetic case is merely an overall sign of the energy spectrum. The distinction between the two cases becomes relevant only when considering the ground state and its low-energy excitations.

Other choices that are sometimes encountered include:

- twists of the closed boundary conditions,
- open boundary conditions with specific boundary Hamiltonians,
- semi-infinite chains.

Some of these boundary conditions are compatible with integrability, others may not.

Boundary conditions have a strong impact on the spectrum: Infinite chains generally have a continuous spectrum while finite chains by have a discrete spectrum by definition. This makes the spectral problem more interesting for finite chains. Here, the closed chains are typically easier to handle than open chains, therefore we shall mainly consider the former.

Spectrum. Consider a finite chain, how to obtain the spectrum?

- Enumerate a basis of $\mathbb{V}^{\otimes L}$, e.g. $|\downarrow \dots \downarrow \downarrow\rangle, |\downarrow \dots \downarrow \uparrow\rangle, \dots$ amounting to 2^L states in total.
- Evaluate \mathcal{H} in this basis as a $2^L \times 2^L$ matrix. This uninspiring task of basic combinatorics leads to a sparse matrix of integer entries.
- Next solve the eigenvalue problem of the Hamiltonian matrix.

The problem is ideally suited for computer algebra:

- One can automatically evaluate the Hamiltonian as a matrix for fairly large L .
- An exact diagonalisation in terms of algebraic numbers is feasible only for small L .
- Numerical evaluation of the eigenvalues allows slightly larger values of L .
- The spectrum is a big mess.
- Eigenvalues appear in multiplets.

Spectrum in Mathematica. Let us present a concise implementation of the XXX Hamiltonian in `Mathematica`.

First, we need to find a way to represent spin chain states. An immediate thought would be to define them as vectors with 2^L components. A drawback of this approach is that one obtains rather abstract and obscure objects which grow exponentially fast with L and which are not so easy to act upon. An alternative and more symbolic approach is to “define” a set of abstract basis vectors and allow for linear combinations. For example, we can represent pure spin chain states by functions whose arguments denote the spin orientations

$$|\uparrow, \uparrow, \downarrow, \uparrow, \downarrow\rangle \rightarrow \text{State}[1, 1, 0, 1, 0]. \quad (3.12)$$

The function `State` is undefined by default, so it remains unevaluated and can be used to represent linear combinations, e.g.

$$10 \text{State}[1, 1, 0, 1, 0] - 5 \text{State}[1, 0, 1, 1, 0]. \quad (3.13)$$

Next we have to represent the Hamiltonian \mathcal{H} through some replacement operator

$$\text{Ham} : \sum * \text{State}[\dots] \rightarrow \sum * \text{State}[\dots]. \quad (3.14)$$

A homogeneous nearest neighbour Hamiltonian can be implemented by the following code:

```
Ham[X_] :=
  X /. Psi_State -> Module[{k, L=Length[Psi]},
    Sum[HamAt[Psi, k, Mod[k+1, L, 1]],
      {k, L}]]];
```

(3.15)

This function replaces (`/.`, `ReplaceAll`) every occurrence of `State` in the argument `X` with the homogeneous action of the kernel `HamAt`. Some notes:

- `Psi_State` symbolises any object `State[...]`, i.e. any object with head `State`.³
- The use of the replacement operator `->` (`RuleDelayed`) as opposed to `->` (`Rule`) is essential because it evaluates the right hand side only after insertion of `Psi`.
- The above definition assumes that the argument `X` is a linear combination of `State` objects. If `X` is not a linear combination of `State` objects, `Ham` does whatever it does (replace objects). Lists, vectors, matrices, nested lists of linear combinations of `State` objects are permissible as arguments: `Ham` will act on each element individually.
- The construct `Module` defines a local variable `k`⁴ and a local variable `L` assigned with the length of the state `Psi`.

The Hamiltonian kernel for the XXX model can be defined as

```
HamAt[Psi_State, k_, l_] :=
  Psi - Permute[Psi, Cycles[{{k,l}}]]];
```

(3.16)

It uses some pre-defined combinatorial methods to implement the permutation of two sites in the symbol `Psi`.

We are now ready to act on states. In order to obtain the complete spectrum we have to enumerate a basis of $\mathbb{V}^{\otimes L}$. As a shortcut, we can employ the binary representation of integers $0, \dots, 2^L - 1$:

```
Basis[L_] :=
  Table[State @@ IntegerDigits[k, 2, L],
    {k, 0, 2^L-1}];
```

(3.17)

Here the operator `@@` (`Apply`) replaces the head of the binary representation of `k` (which is `List`) with `State`. The variable `states` is now a list of pure basis states.

To evaluate the Hamiltonian on the states we can use the following construct:

```
HamMat[states_] :=
  Module[{X=Ham[states]},
    Coefficient[X, #] & /@ states];
```

(3.18)

Some notes:

³Almost all objects in *Mathematica* (except variables and concrete numbers) are *headed lists*. They can be treated much like lists (which are in fact objects with head `List`).

⁴Sometimes using the same variable names as arguments of a `Sum` and elsewhere can lead to undesired interference (depending on the order of evaluation of sub-expressions). To avoid a potential interference it makes sense to make the summation variable local.

- `Ham[states]` evaluates `Ham` on every element of the list `states`. Usually, one would have to explicitly declare this behaviour for the function `Ham` by means of `SetAttributes[Ham, Listable]`. In our case, the definition via a replacement rule automatically implements this desired behaviour.
- The operator `&` (`Function`) represents a *pure function* (a function without a declaration) which returns `Coefficient[X, #]` where `#` is the argument passed to the function. In practice it extracts the coefficient of the argument within `X`.
- The operator `/@` (`Map`) evaluates the above pure function on all elements of the list `states`. This is the matrix representation of `Ham` in the basis `states`.⁵

To finally extract the eigenvalues, generate the Hamiltonian matrix via (remember to substitute or define `L` as a not too large positive integer)

$$\text{emat} = \text{HamMat}[\text{Basis}[L]]; \quad (3.19)$$

and use `Eigenvalues[emat]`, `Eigenvalues[N[emat]]` or `Eigenvalues[N[emat, 20]]`.

Symmetry. The XXX Hamiltonian has a $SU(2)$ Lie group symmetry because the kernel $\mathcal{H}_{k,l}$ is formulated as a manifestly $SU(2)$ invariant operator.

We can set up a representation \mathcal{J}^α , $\alpha = x, y, z$, of the Lie algebra $\mathfrak{su}(2)$ on spin chains

$$\mathcal{J}^\alpha = \sum_{k=1}^L \sigma_k^\alpha. \quad (3.20)$$

- This is a tensor product representation of L spin $1/2$ irreps of $\mathfrak{su}(2)$ given by the Pauli matrices σ_k^α acting on site k .
- The Hamiltonian is invariant

$$[\mathcal{J}^\alpha, \mathcal{H}] = 0. \quad (3.21)$$

- The tensor product is decomposable, for the shortest few chains one finds by the well-known tensor product rules for $\mathfrak{su}(2)$:

$$\begin{aligned} L = 2 &: (1) + (0); \\ L = 3 &: (\frac{3}{2}) + 2(\frac{1}{2}); \\ L = 4 &: (2) + 3(1) + 2(0); \\ L = 5 &: (\frac{5}{2}) + 4(\frac{3}{2}) + 5(\frac{1}{2}); \\ L = 6 &: (3) + 5(2) + 9(1) + 5(0); \\ &\dots \end{aligned} \quad (3.22)$$

Here (s) denotes a finite irrep of spin s .

- Each multiplet has one common eigenvalue.

The spectrum of the closed chain of small length L in units of λ takes a fairly

⁵Potentially, one should `Transpose` the matrix.

simple form:

L	eigenvalue multiplets		
2	$(1) \times 0,$	$(0) \times 4;$	
3	$(\frac{3}{2}) \times 0,$	$2(\frac{1}{2}) \times 3;$	
4	$(2) \times 0,$	$2(1) \times 2,$	$(1) \times 4,$
	$(0) \times 6,$	$(0) \times 2;$	
5	$(\frac{5}{2}) \times 0,$	$2(\frac{3}{2}) \times \frac{1}{2}(5 + \sqrt{5}),$	$2(\frac{3}{2}) \times \frac{1}{2}(5 - \sqrt{5}),$
	$(\frac{1}{2}) \times 4,$	$2(\frac{1}{2}) \times 4 + \sqrt{5},$	$2(\frac{1}{2}) \times 4 - \sqrt{5};$
6	$(3) \times 0,$	$2(2) \times 3,$	$2(2) \times 1,$
	$(2) \times 4,$	$2(1) \times \frac{1}{2}(7 + \sqrt{17}),$	$2(1) \times \frac{1}{2}(7 - \sqrt{17}),$
	$2(1) \times 5,$	$(1) \times 5 + \sqrt{5},$	$(1) \times 5 - \sqrt{5},$
	$(1) \times 2,$	$(0) \times 5 + \sqrt{13},$	$(0) \times 5 - \sqrt{13},$
	$2(0) \times 4,$	$(0) \times 6;$	
		

(3.23)

Note that the $\mathfrak{su}(2)$ eigenvalue multiplets typically appear with an extra multiplicity of 1 or 2. The pairing is largely a consequence of parity symmetry. However, parity is not sufficient to explain all of the pairings. Such extra pairings can be related to integrability.

3.2 Coordinate Bethe Ansatz

The Heisenberg spin chain is quantum integrable. Unfortunately, there is no universal notion of quantum integrability as in the finite-dimensional classical case (Liouville). In particular, it is unclear how to define the number of degrees of freedom in a quantum theory. Let us therefore inspect the consequences of integrability:

Closed Chain Bethe Equations. Consider a set of M algebraic equations (Bethe equations) for the M variables $u_k \in \mathbb{C}$ (Bethe roots)

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j + i}{u_k - u_j - i} \quad \text{for } k = 1, \dots, M. \quad (3.24)$$

Claim: for every eigenstate of \mathcal{H} there is a solution of the above equations with $M \leq L/2$ distinct Bethe roots u_k .⁶ The energy eigenvalue of this state can be read off easily

$$E = \lambda \sum_{k=1}^M \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}} \right). \quad (3.25)$$

⁶There are some subtleties related to the $SU(2)$ symmetry for the XXX model, and one has to pay attention to Bethe roots at ∞ , $\pm \frac{i}{2}$ and Bethe roots separated by i .

Example: $L = 6$, $M = 3$ (corresponding to a spin singlet)

$$u_{1,2} = \pm \sqrt{-\frac{5}{12} + \frac{\sqrt{13}}{6}}, \quad u_3 = 0, \quad E = \lambda(5 + \sqrt{13}). \quad (3.26)$$

Benefits:

- We have transformed a problem of linear algebra directly to algebraic equations. We can thus skip combinatorics and characteristic polynomials.
- We can use the Bethe equations efficiently for approximations at large L and M . For example, the anti-ferromagnetic ground state can be approximated at large L in which case the Bethe equations turn into integral equations.

In the following we shall derive the Bethe equations from the coordinate Bethe ansatz.

Vacuum State. Start with a simple state, the *ferromagnetic vacuum*

$$|0\rangle = |\downarrow\downarrow\dots\downarrow\rangle. \quad (3.27)$$

By construction this state has zero energy

$$\mathcal{H}_{k,k+1}|0\rangle = \mathcal{I}_{k,k+1}|0\rangle - \mathcal{P}_{k,k+1}|0\rangle = |0\rangle - |0\rangle = 0. \quad (3.28)$$

Therefore $\mathcal{H}|0\rangle = 0$ and the ground state energy is zero

$$E = 0. \quad (3.29)$$

This solves the problem for $M = 0$ corresponding to the multiplet $(L/2)$.

Magnon States. Now flip one spin at site k

$$|k\rangle = |\downarrow\dots\downarrow\overset{k}{\uparrow}\downarrow\dots\downarrow\rangle. \quad (3.30)$$

These states enumerated by k form a closed sector under the Hamiltonian due to conservation of the z -component of spin \mathcal{J}^z .

How to obtain eigenstates of \mathcal{H} ? Note that the Hamiltonian is homogeneous and commutes with a shift of the chain by one unit.⁷ We can thus look for simultaneous eigenstates of the Hamiltonian and the shift operator. Momentum eigenstates are plane waves⁸

$$|p\rangle = \sum_k e^{ipk} |k\rangle. \quad (3.31)$$

This state is called a *magnon state*. It can be viewed as a particle excitation⁹ of the above vacuum state.

⁷The lattice shift is a discrete version of the momentum generator.

⁸The notation is slightly ambiguous, but it should become clear from the context whether $|*\rangle$ refers to a position eigenstate $|k\rangle$ or a momentum eigenstate $|p\rangle$.

⁹Here the notion of particle is an object which carries an individual momentum p .

Since there is a unique state with a given momentum p , it must already be an energy eigenstate. We can now act with \mathcal{H} on $|p\rangle$ and obtain (after a shift of summation variable to match the states on the r.h.s.)

$$\begin{aligned}\mathcal{H}|p\rangle &= \lambda \sum_k e^{ipk} \left(\overbrace{|k\rangle - |k-1\rangle}^{H_{k-1,k}} + \overbrace{|k\rangle - |k+1\rangle}^{H_{k,k+1}} \right) \\ &= \lambda \sum_k e^{ipk} (1 - e^{ip} + 1 - e^{-ip}) |k\rangle \\ &= e(p) |p\rangle\end{aligned}\tag{3.32}$$

with the *magnon dispersion relation*

$$e(p) = 2\lambda(1 - \cos p) = 4\lambda \sin^2(\frac{1}{2}p).\tag{3.33}$$

For a closed chain, the momentum is quantised by the periodic boundary conditions to

$$p = \frac{2\pi in}{L}, \quad \text{where } n = 0, \dots, L-1.\tag{3.34}$$

For an infinite chain p is a continuous parameter. Note that in both cases the momentum is defined only modulo 2π because the position k is sampled only at the discrete lattice positions. A shift by 2π corresponds to a change of Brillouin zone which leaves the eigenstate unchanged.

This solves the problem for $M = 1$ corresponding to the multiplets $(L/2 - 1)$.¹⁰

Scattering Factor. We continue with states with two spin flips

$$|k < l\rangle = |\downarrow \dots \downarrow \overset{k}{\uparrow} \downarrow \dots \downarrow \overset{l}{\uparrow} \downarrow \dots \downarrow\rangle.\tag{3.35}$$

Here we make the assumption that $k < l$. Again, these states form a closed sector for the Hamiltonian, and we wish to construct eigenstates.

When the spin flips are well-separated we can treat the state as the combination of two individual magnons. The nearest neighbour Hamiltonian will hardly ever see both spin flips at the same time, therefore we can make an ansatz for eigenstates of the form

$$|p < q\rangle = \sum_{k < l = -\infty}^{+\infty} e^{ipk+iql} |k < l\rangle.\tag{3.36}$$

Some comments:

- This state has overall momentum $P = p + q$.
- The momenta of the individual spin flips are not well-defined because their wave functions do not extend over the whole chain, but are constrained by an ordering of the spin flips. Nevertheless we can use p and q as labels for a particular state.
- The notation $|p < q\rangle$ is not meant to imply that p is less than q , but rather that the magnon with momentum p is to the right of the magnon with momentum q .

¹⁰The state with $p = 0$ belongs to the multiplet $(L/2)$ discussed in the context of the vacuum.

- For the time being we shall only consider an infinite chain so that we do not have to worry about boundary conditions.

By construction, each of these states is almost an eigenstate with eigenvalue

$$E = e(p) + e(q). \quad (3.37)$$

Acting with the combination $\mathcal{H} - e(p) - e(q)$ on $|p < q\rangle$ yields

$$(e^{ip+iq} - 2e^{iq} + 1) \sum_{k=-\infty}^{+\infty} e^{i(p+q)k} |k < k+1\rangle. \quad (3.38)$$

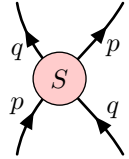
Only a contact term $\sum_k e^{i(p+q)k} |k < k+1\rangle$ remains and violates the eigenstate condition. Since this state is symmetric in p and q we can act on the state $|q < p\rangle$ with the magnon momenta interchanged and obtain a proportional term

$$(e^{ip+iq} - 2e^{ip} + 1) \sum_{k=-\infty}^{+\infty} e^{i(p+q)k} |k < k+1\rangle. \quad (3.39)$$

We can now patch together the two partial wave functions and construct an exact eigenstate¹¹

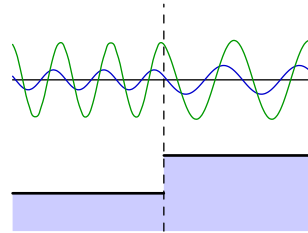
$$|p, q\rangle = |p < q\rangle + S(p, q)|q < p\rangle \quad (3.40)$$

with a *scattering factor* S

$$S(p, q) = -\frac{e^{ip+iq} - 2e^{iq} + 1}{e^{ip+iq} - 2e^{ip} + 1}. \quad (3.41)$$


The scattering factor is analogous to the scattering factor or scattering matrix in QM and QFT.¹²

The process of patching together two wave functions is analogous to the construction of quantum mechanical wave functions at a potential step.



$$(3.42)$$

In our context, the distance of the magnons is the relevant position variable and the potential step is at the minimum distance of 1 lattice site.

¹¹We pay no attention to the overall normalisation of the state. Therefore the state $|p, q\rangle$ is independent of the order of p and q (up to normalisation).

¹²The setup for these two objects is slightly different: The ordinary scattering factor relates ingoing and outgoing states where each particle has a well-defined momentum in the distant past and distant future. The scattering factor of the Bethe ansatz relates two partial wave functions with different ordering of the individual particles. The latter is something that works only with one spatial dimension. Therefore an ordinary scattering factor corresponds to a time-like process whereas the scattering factor in the Bethe ansatz corresponds to a space-like process.

Factorised Scattering. Before considering closed chains, let us have a look at three-magnon states. There are $6 = 3!$ asymptotic regions for magnons which carry one momentum p_k each. A useful ansatz for an eigenstate is the so-called *Bethe ansatz*

$$\begin{aligned} |p_1, p_2, p_3\rangle = & |p_1 < p_2 < p_3\rangle + S_{12}S_{13}S_{23}|p_3 < p_2 < p_1\rangle \\ & + S_{12}|p_2 < p_1 < p_3\rangle + S_{13}S_{23}|p_3 < p_1 < p_2\rangle \\ & + S_{23}|p_1 < p_3 < p_2\rangle + S_{12}S_{13}|p_2 < p_3 < p_1\rangle. \end{aligned} \quad (3.43)$$

All pairwise contact terms vanish by construction due to the choice of appropriate pairwise scattering factors between any two partial wave functions. There could in principle be a triple contact term

$$(\mathcal{H} - E)|p_1, p_2, p_3\rangle \sim \sum_k e^{iP_k} |k < k + 1 < k + 2\rangle. \quad (3.44)$$

Due to a miracle this contact term is absent without further ado. This miracle is called integrability.¹³ It works analogously for any number of magnons as we shall discuss below. We only need the two-magnon scattering factor to construct arbitrary states.

In other words, scattering of more than two magnons factorises into a sequence of pairwise magnon scattering processes, for example:

(3.45)

For pairwise scattering, conservation of total momentum and total energy ensures that the individual momenta are not deformed but merely exchanged. Therefore, in factorised scattering processes, the particle momenta are only ever permuted. Conversely, if the particle momenta are merely permuted by scattering, the above ansatz for the complete wave function must be complete, and scattering factorises.

Factorised scattering means that there is no elementary scattering factor for three or more magnons. Such a factor would lead to a substantially different behaviour in that it must deform the momenta of the particles continuously. This is an option for three or more particles because conservation of total momentum and total energy are not sufficient to guarantee the conservation of all individual momenta. Conservation of additional commuting charges (which can be measured simultaneously) excludes deformations of the kinematical configuration, and therefore implies factorised scattering.

¹³The absence of the contact term is potentially a consequence of the pairwise nature of the Hamiltonian. This is not in contradiction with the fact that most models with nearest neighbour Hamiltonians are non-integrable. Generically these models have a pairwise scattering matrix which is not consistent with the assumption of factorised scattering in which case the above ansatz has leftover pairwise contact terms.

Solution of the Infinite Chain. We have found an explicit and exact solution for the eigenstates of the infinite chain with an arbitrary number of magnons.

$$\begin{aligned}
|0\rangle &= |\downarrow \dots \downarrow\rangle, & E &= 0, \\
|p\rangle &= \sum_k e^{ipk} |\dots \overset{k}{\uparrow} \dots\rangle, & E &= e(p), \\
|p, q\rangle &= |p < q\rangle + S(p, q) |q < p\rangle, & E &= e(p) + e(q), \\
|p_k\rangle &= \sum_{\pi \in S_M} S_\pi |p_{\pi(1)} < \dots < p_{\pi(M)}\rangle, & E &= \sum_k e(p_k)
\end{aligned} \tag{3.46}$$

The momenta p_k are arbitrary numbers, therefore the spectrum is continuous.

Note:

- The ordering of the p_k does not matter: magnons are identical particles.
- The momenta p_k are defined modulo 2π : they move on a lattice.
- The momenta should be real for wave functions to be normalisable (in the ordinary sense of plane waves).
- For two identical momenta

$$S(p, p) = -1 \implies |p, p, \dots\rangle = 0. \tag{3.47}$$

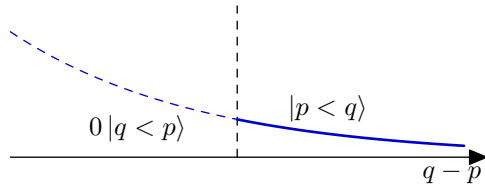
The value of the scattering factor and the exclusion principle indicates that the particles obey Fermi statistics. The XXX model on the infinite chain is equivalent to *free fermions* on a one-dimensional lattice!

- Zero-momentum particles are special:

$$S(p, 0) = 1, \quad e(0) = 0. \tag{3.48}$$

They behave as free bosons which do not interact with any of the other particles. They represent the $\mathfrak{su}(2)$ ladder operators which allow to walk between the states of a $SU(2)$ multiplet.

- Whenever $S(p, q) = 0, \infty$, some partial wave functions of a state $|p, q, \dots\rangle$ have coefficient 0.¹⁴ This allows the involved momenta to be complex under certain conditions. For this to happen, the exponentially growing regions of the wave function, which would normally make the state non-normalisable, must be removed by a zero coefficient.



$$\tag{3.49}$$

Such states are called *bound states*. They can be viewed as different types of particle excitations with one independent (real) momentum and a different dispersion relation. Bound states made from more than two magnons also exist.

¹⁴One has to arrange the overall normalisation such that none of the relevant scattering factors is ∞ .

Periodicity and Bethe Equations. We know how to solve the infinite XXX chain, but we would like to understand the spectrum of the finite closed chain. To this end, we can compare states of the closed chain with periodic states of the infinite chain. There are at least two conceivable notions of periodic states:

- states with periodic excitations

$$|\dots, k_1 - L, k_1, k_1 + L, \dots, k_2 - L, k_2, k_2 + L, \dots\rangle. \quad (3.50)$$

- states with a periodic wave function

$$\langle k, \dots | \Psi \rangle = \langle k + L, \dots | \Psi \rangle. \quad (3.51)$$

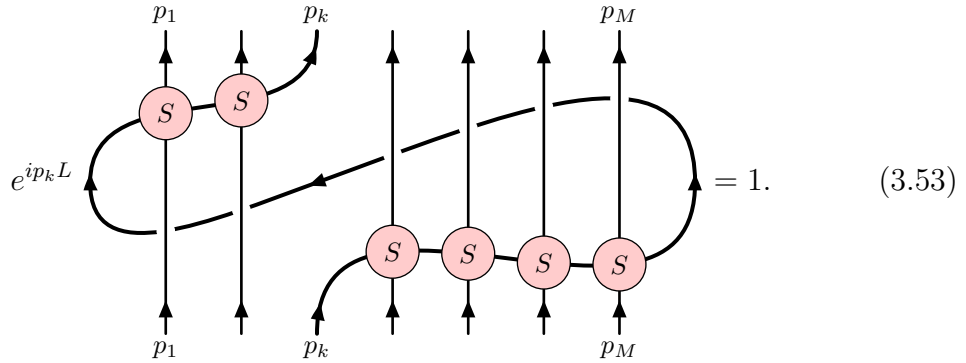
The advantage of the latter point of view is that it requires only finitely many excitations. Let us therefore continue along these lines:

- Consider a position space configuration where the magnons are separated by less than L sites.¹⁵
- Focus on the leftmost excitation and pay attention to how the wave function of an eigenstate evolves as this excitation is shifted towards the right.
- Moving the excitation by L sites generates a factor of $e^{ip_k L}$ by construction.
- Along the way, it will move past all the other excitations and pick up a factor of $S(p_k, p_j)$ for each permutation.¹⁶
- The eigenstate is periodic if all the phase factors multiply to 1.
- Such a periodic wave function can be lifted to the wave function on a closed chain.¹⁷

This leads to the *Bethe equations* for a closed chain

$$e^{ip_k L} \prod_{\substack{j=1 \\ j \neq k}}^M S(p_k, p_j) = 1, \quad \text{for all } k = 1, \dots, M. \quad (3.52)$$

Graphically, the Bethe equations can be represented as follows:



¹⁵In fact, the states are not actually periodic when the magnons are spread along the chain by more than L sites.

¹⁶The wave function changes rapidly at a single site, there is no interaction of the magnons at a distance. This is a crucial insight to make this construction exact.

¹⁷The above restriction to a range of at most L sites is crucial. When moving the magnon further towards the right, no further scattering factor can be picked up, and the wave function does not have a different periodic behaviour.

They amount to one equation for each unknown variable p_k . This effectively quantises the spectrum. A simple consistency requirement for the closed chain already leads to a discrete set of solutions.

The total energy and total momentum of a solution can be read off from the set of magnon momenta p_k

$$E = \sum_{k=1}^M e(p_k), \quad P = \sum_{k=1}^M p_k. \quad (3.54)$$

One can simply derive a useful statement on P by multiplying all Bethe equations, namely

$$e^{iPL} = 1. \quad (3.55)$$

This relationship follows from triviality of an overall shift by L sites where e^{iP} is the eigenvalue of the cyclic shift operator.

Rapidities. It is convenient to introduce a different set of variables u_k instead of the momenta p_k .

$$p_k = 2 \operatorname{arccot} 2u_k, \quad u_k = \frac{1}{2} \cot \frac{1}{2} p_k. \quad (3.56)$$

The scattering factor simplifies to a rational function

$$S(u, v) = \frac{u - v - i}{u - v + i}. \quad (3.57)$$

The Bethe equations then take the form introduced earlier

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j + i}{u_k - u_j - i} \quad \text{for } k = 1, \dots, M. \quad (3.58)$$

and the energy and momentum eigenvalues are obtained via

$$e^{iP} = \prod_{k=1}^M \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}, \quad E = \lambda \sum_{k=1}^M \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}} \right). \quad (3.59)$$

Let us mention some special points and configurations:

- The u_k are real or form complex conjugate pairs.¹⁸
- All u_k must be distinct except for the special value $u_k = \infty$ which can appear several times.
- The $\mathfrak{su}(2)$ ladder operators at $p_k = 0$ correspond to $u_k = \infty$.
- The special values $u_k = \pm \frac{i}{2}$ where $p_k = \infty$ which typically do not appear in physically relevant solutions. However, some relevant singular solutions exist.
- There is no analog of the periodicity of the p_k for the u_k .
- We should restrict to $M \leq \frac{1}{2}L$; the other states with $M > \frac{1}{2}L$ are represented via a collection of $u_k = \infty$ added to a solution with $M \leq \frac{1}{2}L$.
- Bound states with $S(u_k, u_j) = 0, \infty$ are obtained for the simple condition $u_k = u_j \pm i$. Higher bound states correspond to so-called *Bethe strings* $u_k = u_0 + ik$.

¹⁸Normalisability is not an issue for finite chains.

3.3 Generalisation

Open Chains. Consider an open chain with Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{L-1} \mathcal{H}_{k,k+1}. \quad (3.60)$$

To quantify the effect of the boundaries, consider a semi-infinite chain starting at site $k = 1$. Act with $\mathcal{H} - e(p)$ on a one-magnon state $|+p\rangle$.

$$(\mathcal{H} - e(p))|+p\rangle = (1 - e^{+ip})|1\rangle. \quad (3.61)$$

As for the two-magnon state, there is a residual term located at the boundary. This term can be compensated by another partial eigenstate with equal energy $e(\bar{p}) = e(p)$, namely $\bar{p} = -p$.

$$(\mathcal{H} - e(p))|-p\rangle = (1 - e^{-ip})|1\rangle. \quad (3.62)$$

Now combine the states into an exact eigenstate^{19 20}

$$|p\rangle = e^{-ip}|+p\rangle + e^{+ip}K_L(-p)|-p\rangle \quad (3.63)$$

with the boundary scattering factor

$$K_L(-p) = -e^{-2ip} \frac{1 - e^{+ip}}{1 - e^{-ip}} = e^{-ip}. \quad (3.64)$$

Similarly, one can construct exact eigenstates for a semi-infinite chain ending at site $k = L$

$$|p\rangle = e^{-ipL}|+p\rangle + e^{+ipL}K_R(+p)|-p\rangle \quad (3.65)$$

with boundary scattering factor

$$K_R(+p) = e^{+ip}. \quad (3.66)$$

Compatibility of both boundaries leads to a set of Bethe equations for the open chain

$$\frac{\exp(i(L-1)(+p_k))}{\exp(i(L-1)(-p_k))} \frac{K_R(+p_k)}{K_L(-p_k)} \prod_{\substack{j=1 \\ j \neq k}}^M \frac{S(+p_k, p_j)}{S(-p_k, p_j)} = 1, \quad (3.67)$$

or in a graphical representation:

$$\frac{e^{+ip_k(L-1)}}{e^{-ip_k(L-1)}} K_L \dots K_R = 1. \quad (3.68)$$

¹⁹The factors of $e^{\pm ip}$ were inserted to compensate for the plane wave factor at site $k = 1$.

²⁰Up to normalisation, the exact eigenstates are invariant under flipping the sign of the momentum p because they are a superposition of ingoing and outgoing waves.

Note that these equations are invariant under flipping the sign of any momentum $p_j \rightarrow -p_j$. Flipping the sign of p_k inverts the equation.

The Bethe equations in rational form read

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^{2L} = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j + i}{u_k - u_j - i} \frac{u_k + u_j + i}{u_k + u_j - i}. \quad (3.69)$$

Modified boundaries lead to some additional factors in the equations.

Bethe Equations for the XXZ Model. The XXX model is part a larger XXZ family of integrable models which are solvable by the above Bethe ansatz.²¹ Strictly speaking the XXZ model is the model defined above. However, we can add a few parameters while preserving the features of the original model²²

$$\begin{aligned} \mathcal{H}_{k,k+1} = & \alpha_1(1 \otimes 1) + \alpha_2(\sigma^z \otimes 1) + \alpha_3(1 \otimes \sigma^z) + \alpha_2(\sigma^z \otimes \sigma^z) \\ & + \alpha_5(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) + i\alpha_6(\sigma^x \otimes \sigma^y + \sigma^y \otimes \sigma^x). \end{aligned} \quad (3.70)$$

The 6 free parameter have the following meaning:

- one overall shift of energies: $\delta\alpha_1$,
- one trivial deformation for closed chains: $\delta\alpha_2 = -\delta\alpha_3$,
- one shift proportional to \mathcal{J}^z : $\delta\alpha_2 = +\delta\alpha_3$,
- one overall scaling of energies: $\delta\alpha_k = \alpha_k\delta\beta$,
- one quantum deformation parameter \hbar also known as $q = e^{i\hbar}$ and the anisotropy $\Delta = \frac{1}{2}(q + q^{-1})$,
- one magnetic flux parameter ρ .

The resulting Bethe equations for closed chains read

$$\left(\frac{\sin \hbar(u_k + \frac{i}{2})}{\sin \hbar(u_k - \frac{i}{2})}\right)^L e^{i\rho L} = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{\sin \hbar(u_k - u_j + i)}{\sin \hbar(u_k - u_j - i)}. \quad (3.71)$$

These Bethe equations are called *trigonometric* as opposed to the *rational* Bethe equations for the XXX model.²³ The total momentum and energy are given by

$$P = \sum_{k=1}^M p(u_k), \quad E = \gamma_1 L + \gamma_2 M + \gamma_3 \sum_{k=1}^M e(u_k) \quad (3.72)$$

²¹The latter is part of the even larger XYZ family, but its solution requires more advanced techniques because there is no U(1) symmetry to preserve the number of magnons.

²²This is in fact the most general nearest neighbour Hamiltonian which commutes with $\mathcal{J}^z = \sum_k \sigma_k^z$.

²³Both sets of Bethe equations can be written in either rational or trigonometric form with a suitable choice of variables, e.g. $z_k = \exp(i\hbar u_k)$ for XXZ. The distinguished set of variables, however, is where u_j appears only in the combination $u_j - u_k$. Using these variables the Bethe equations are rational and trigonometric for XXX and XXZ, respectively.

with

$$e^{ip(u)} = \frac{\sin \hbar(u + \frac{i}{2})}{\sin \hbar(u - \frac{i}{2})}, \quad e(u) = p'(u). \quad (3.73)$$

Evidently, these equations reduce to the rational case in the limit $\hbar \rightarrow 0$.

XXX model with Higher Spin. We can also use a different Hilbert space for the spin chain, for example a spin $s = 1$ representation spanned by three states $|0\rangle$, $|1\rangle$ and $|2\rangle$ corresponding to spin up, spin zero and spin down. The so-called XXX₁ Hamiltonian has SU(2) symmetry, in particular it preserves \mathcal{J}^z . Therefore, the Hamiltonian kernel takes a block-diagonal form in the basis E

$$\mathcal{H}_{k,k+1} = \begin{pmatrix} * & & & & \\ & * & * & & \\ & * & * & & \\ & & & * & * & * \\ & & & * & * & * \\ & & & * & * & * \\ & & & & * & * \\ & & & & * & * \\ & & & & & * \end{pmatrix}, \quad E = \begin{pmatrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |20\rangle \\ |11\rangle \\ |02\rangle \\ |21\rangle \\ |12\rangle \\ |22\rangle \end{pmatrix}. \quad (3.74)$$

We do not reproduce the coefficients because they do not add a qualitative insight.

The above Bethe ansatz works with small alterations:

- vacuum:

$$|0\rangle = |0 \dots 0\rangle. \quad (3.75)$$

- one-magnon states:

$$|p\rangle = \sum_k e^{ipk} |\dots \overset{k}{1} \dots\rangle. \quad (3.76)$$

- two-magnon states:

$$\begin{aligned} |p < q\rangle &= \sum_k e^{ipk+iql} |\dots \overset{k}{1} \dots \overset{l}{1} \dots\rangle, \\ |p; 2\rangle &= \sum_k e^{ipk} |\dots \overset{k}{2} \dots\rangle. \end{aligned} \quad (3.77)$$

The action of the Hamiltonian on partial eigenstates now yields some additional terms

$$\begin{aligned} (\mathcal{H} - E)|p < q\rangle &= \sum_k e^{i(p+q)k} (*|\dots \overset{k}{11} \dots\rangle + *|\dots \overset{k}{2} \dots\rangle), \\ (\mathcal{H} - E)|p; 2\rangle &= \sum_k e^{ipk} (*|\dots \overset{k}{11} \dots\rangle + *|\dots \overset{k}{2} \dots\rangle). \end{aligned} \quad (3.78)$$

The scattering ansatz needs to be supplemented in order to compensate them appropriately.

$$|p, q\rangle = |p < q\rangle + S|q < p\rangle + C|p + q; 2\rangle. \quad (3.79)$$

To construct the exact eigenstate we now have to solve two linear equations. The coefficient S is the scattering factor which is relevant for IR physics. The contact

term C is important for the solution, but it merely describes the UV physics of the eigenstate.²⁴

The resulting Bethe equations for a closed chain read

$$\left(\frac{u_k + i}{u_k - i}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j + i}{u_k - u_j - i}, \quad e^{ip} = \frac{u + i}{u - i}, \quad e(u) = p'(u). \quad (3.80)$$

Note that the Bethe equations are almost the same up to a different prefactor of i on the l.h.s. of the Bethe equations and likewise in the definition of the magnon momentum.

The generalisation to arbitrary spin s representations at each site is evident (and correct)

$$\left(\frac{u_k + is}{u_k - is}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j + i}{u_k - u_j - i}, \quad e^{ip} = \frac{u + is}{u - is}. \quad (3.81)$$

The corresponding model is called the XXX_s model.

Bethe Ansatz at Higher Rank. Generalisation of the XXX model to higher-rank groups exist. For example, consider a chain with $\text{SU}(N)$ symmetry and spins in the fundamental representation

$$\mathbb{V} = \mathbb{C}^N, \quad |1\rangle, \dots, |N\rangle \in \mathbb{V}. \quad (3.82)$$

An integrable nearest neighbour Hamiltonian is given by the kernel

$$\mathcal{H}_{k,k+1} = \mathcal{I}_{k,k+1} - \mathcal{P}_{k,k+1}. \quad (3.83)$$

More explicitly, this kernel acts as follows

$$\mathcal{H}|ab\rangle = |ab\rangle - |ba\rangle. \quad (3.84)$$

We can again perform the Bethe ansatz:

- vacuum:

$$|0\rangle = |1 \dots 1\rangle. \quad (3.85)$$

- there are now $N - 1$ flavours of one-magnon states labelled by $a = 2, \dots, N$

$$|p, a\rangle = \sum_k e^{ipk} |\dots \overset{k}{a} \dots\rangle. \quad (3.86)$$

²⁴The term $|p + 2; 2\rangle$ should be viewed as a contribution when both magnons reside on a single site. We did not have to consider such terms before because for spin 1/2 a single site can only be excited once.

- To accommodate for the various combinations of magnon flavours, we need a *scattering matrix*²⁵ instead of a scattering factor for the definition of two-magnon states

$$|(p, a), (q, b)\rangle = |(p, a) \langle (q, b)\rangle + \sum_{c,d=2}^N S_{ab}^{cd}(p, q) |(q, d) \langle (p, c)\rangle. \quad (3.87)$$

The S-matrix may again be represented graphically as follows:



The scattering matrix is a new feature for models based on a higher-rank algebra.

- The matrix can be computed as before by matching all asymptotic regions. In our case, one finds

$$S_{ab}^{cd}(u, v) = \frac{(u - v)\delta_a^c \delta_b^d + i\delta_a^d \delta_b^c}{u - v - i}. \quad (3.89)$$

- It preserves the residual $SU(N - 1)$ of the magnons on the vacuum state.
- For $u \rightarrow \infty$ or $v \rightarrow \infty$ it is trivial

$$S_{ab}^{cd}(\infty, v) = S_{ab}^{cd}(u, \infty) = \delta_a^c \delta_b^d. \quad (3.90)$$

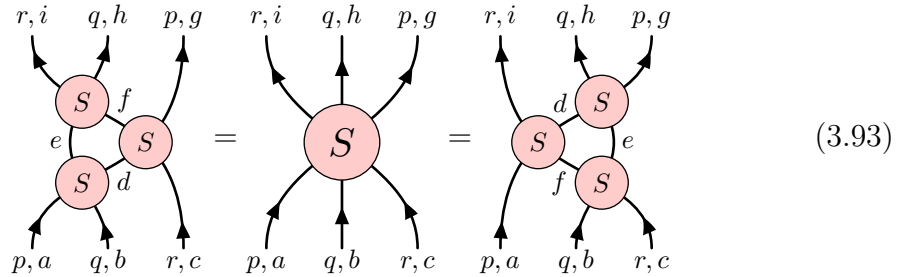
- For equal rapidities it reads

$$S_{ab}^{cd}(u, u) = -\delta_a^d \delta_b^c. \quad (3.91)$$

- It satisfies the *Yang-Baxter equation* which guarantees that states of factorised scattering can be defined consistently

$$S_{ab}^{de}(p, q) S_{dc}^{gf}(p, r) S_{ef}^{hi}(q, r) = S_{bc}^{ef}(q, r) S_{af}^{di}(p, r) S_{de}^{gh}(p, q). \quad (3.92)$$

The flow of indices is best explained using a figure:



An abbreviated version of the formal expression reads

$$S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12}. \quad (3.94)$$

This equation is a central relation for all quantum integrable systems.

²⁵More precisely it is a tensor of rank 4, but when acting on two-magnon states it can be viewed as a matrix.

Nested Bethe Ansatz. The S-matrix now changes the flavour of the particles which are scattering. We thus cannot (easily) set up a consistency equation for periodic wave functions. We would like to “diagonalise” the S-matrix. However, there is no universal method to diagonalise a tensor, but this procedure has to be carefully designed for the problem in question:

- Step 1: Consider a new vacuum state

$$|0\rangle^2 = |2_1 2_2 \dots 2_M\rangle := |(p_1, 2), \dots, (p_M, 2)\rangle. \quad (3.95)$$

The S-matrix is applied easily to this state because scattering is automatically a plain factor $S_{22}^{22}(p, q)$.

- Step 2: Introduce a new types of excitations on the above vacuum

$$|(u, a)\rangle^2 = \sum_{k=1}^M \psi_k(u) |2_1 \dots 2_{k-1} a_k 2_{k+1} \dots 2_M\rangle. \quad (3.96)$$

There are now $N - 2$ types of excitations labelled by $a = 3, \dots, N$. The new wave function $\psi_k(u)$ is not a plane wave because the vacuum state $|0\rangle^2$ is not homogeneous. It must be carefully chosen to enable an easy construction of scattering states and thus it depends on all the underlying magnon momenta p_k . We refrain from presenting the details.

- Step 3: Constructing states with two new excitations lead to a new S-matrix S^2 with $(N - 2)^4$ components. This S-matrix has precisely the same form as the previous one but with fewer components.

This procedure is reminiscent of the Bethe ansatz. In terms of states and excitations, we have achieved the following:

spins	magnons	excitations	
$ 1\rangle$	$ 1\rangle$	$ 1\rangle$	
$ 2\rangle$	$ 1\rangle \rightarrow 2\rangle$	$ 1\rangle \rightarrow 2\rangle$	(3.97)
$ 3\rangle$	$ 1\rangle \rightarrow 3\rangle$	$ 2\rangle \rightarrow 3\rangle$	
\vdots	\vdots	\vdots	
$ N\rangle$	$ 1\rangle \rightarrow N\rangle$	$ 2\rangle \rightarrow N\rangle$	

The Bethe ansatz singles out the vacuum state $|1\rangle$ and converts all other spin states to magnon excitations $|1\rangle \rightarrow |a\rangle$ with $a = 2, \dots, N$. The next step singles out one of the magnon excitations $|1\rangle \rightarrow |2\rangle$ and declares it as a new vacuum. The remaining magnons are obtained as new excitations $|2\rangle \rightarrow |a\rangle$ of the new vacuum with $a = 3, \dots, N$. The procedure, called the *nested Bethe ansatz* can be iterated $N - 1$ times in total. At the end we are left with

- the vacuum state $|1\rangle$,
- the magnon excitation $|1\rangle \rightarrow |2\rangle$,
- $N - 2$ higher excitations $|a - 1\rangle \rightarrow |a\rangle$ with $a = 3, \dots, N$.

Importantly, these interactions now all scatter diagonally, so the scattering matrix has been disintegrated into a collection of scattering factors $S^{a,b}(u^a, v^b)$

$$(3.98)$$

There is no mixing between the various flavours of excitations. It is in fact excluded by conservation of charges of the excitations.

For a given set of excitations, one can construct an eigenstate on the infinite chain. These arise as a sum over all admissible distributions of the excitations. In each distribution we must stack the excitations on the vacuum sites as towers without gaps

$$|4, 3, 1, 4, 1, 1, 2\rangle \longrightarrow \begin{array}{cccccc} 4(u_1^4) & & 4(u_2^4) & & & \\ 3(u_1^3) & 3(u_2^3) & 3(u_3^3) & & & \\ 2(u_1^2) & 2(u_2^2) & 2(u_3^2) & & 2(u_4^2) & \\ |1\rangle & |1\rangle & |1\rangle & |1\rangle & |1\rangle & |1\rangle \end{array} \quad (3.99)$$

The relative phase factors between two distributions are determined by hopping rules: There is a factor for moving one excitation on top of another excitation from the left.

$$b(v^b) \xrightarrow{F^{a,b}(u^a, v^b)} \begin{array}{c} b(v^b) \\ a(u^a) \end{array} \quad (3.100)$$

This factor $F^{a,b}(u^a, v^b)$ depends on the flavours of the excitations and on their rapidities. Note that moving two excitations past each other yields their scattering factor

$$S^{a,b}(u^a, v^b) = \frac{F^{a,b}(u^a, v^b)}{F^{b,a}(v^b, u^a)}. \quad (3.101)$$

Bethe Equations for Higher Rank. For $SU(N)$, the elements of the diagonalised scattering matrix simply read

$$\begin{aligned} S^{a,a}(u^a, v^a) &= \frac{u^a - v^a - i}{u^a - v^a + i}, \\ S^{a,a\pm 1}(u^a, v^{a\pm 1}) &= \frac{u^a - v^{a\pm 1} + \frac{i}{2}}{u^a - v^{a\pm 1} - \frac{i}{2}}, \\ S^{a,b}(u^a, v^b) &= 1 \quad \text{for } |a - b| > 1. \end{aligned} \quad (3.102)$$

It is straightforward to set up the Bethe equations for a closed chain. The Bethe equations for the level-1 magnons read

$$\left(\frac{u_k^1 + \frac{i}{2}}{u_k^1 - \frac{i}{2}} \right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{M^1} \frac{u_k^1 - u_j^1 + i}{u_k^1 - u_j^1 - i} \prod_{j=1}^{M^2} \frac{u_k^1 - u_j^2 - \frac{i}{2}}{u_k^1 - u_j^2 + \frac{i}{2}}. \quad (3.103)$$

The Bethe equations for higher-level excitations take the form

$$1 = \prod_{j=1}^{M^{a-1}} \frac{u_k^a - u_j^{a-1} - \frac{i}{2}}{u_k^a - u_j^{a-1} + \frac{i}{2}} \prod_{\substack{j=1 \\ j \neq k}}^{M^a} \frac{u_k^a - u_j^a + i}{u_k^a - u_j^a - i} \prod_{j=1}^{M^{a+1}} \frac{u_k^a - u_j^{a+1} - \frac{i}{2}}{u_k^a - u_j^{a+1} + \frac{i}{2}}, \quad (3.104)$$

and the top-level equations read

$$1 = \prod_{j=1}^{M^{N-2}} \frac{u_k^{N-1} - u_j^{N-2} - \frac{i}{2}}{u_k^{N-1} - u_j^{N-2} + \frac{i}{2}} \prod_{\substack{j=1 \\ j \neq k}}^{M^{N-1}} \frac{u_k^{N-1} - u_j^{N-1} + i}{u_k^{N-1} - u_j^{N-1} - i}. \quad (3.105)$$

The total momentum and energy are expressed as

$$e^{iP} = \prod_{k=1}^{M^1} \frac{u_k^1 + \frac{i}{2}}{u_k^1 - \frac{i}{2}}, \quad E = \sum_{k=1}^{M^1} \left(\frac{i}{u_k^1 + \frac{i}{2}} - \frac{i}{u_k^1 - \frac{i}{2}} \right). \quad (3.106)$$

Inspecting these equations leads to the following generalisation to arbitrary simple Lie (super)algebras

- The interactions of the Bethe roots (r.h.s. of the Bethe equations) follow the symmetric Cartan matrix DA of the underlying group.²⁶ The latter is directly encoded into the Dynkin diagram of the corresponding algebra. For example, for $A_{N-1} \simeq \text{SU}(N)$:

$$\begin{array}{ccccccc} \bigcirc & \text{---} & \bigcirc & \text{---} & \bigcirc & \text{---} & \dots & \text{---} & \bigcirc & \text{---} & \bigcirc \\ 1 & & 2 & & 3 & & & & N-2 & & N-1 \end{array} \quad (3.107)$$

- The momentum, energy and propagation of the Bethe roots (l.h.s. of the Bethe equations) follow the Dynkin labels of the underlying spin representation.

These equations have all of the generalisations discussed above: trigonometric deformations, introduction of magnetic fluxes, open chains, higher representations. One can also work make the spin chain inhomogeneous while preserving integrability. This can be achieved by a non-homogeneous (and typically non-local) Hamiltonian and by using site-dependent spin representations.

²⁶For the simply-laced groups A, D, E the Cartan matrix A is symmetric and $D = 1$. For the other groups $D = \text{diag}(\dots)$ makes the asymmetric Cartan matrix A symmetric in the product DA .

4 Quantum Integrability

Next we would like to discuss a formalism to deal with a large class of quantum integrable systems.

4.1 R-Matrix Formalism

In the above nested Bethe ansatz we have

- started with an $SU(N)$ fundamental spin chain,
- reduced it to magnons with $SU(N - 1)$ residual symmetry,
- reduced it further to excitations with $SU(N - 2)$ residual symmetry,
- ...
- obtained a final set of excitations with no residual symmetry.

An interesting observation is that in every step we obtained a scattering matrix of the same form

$$S_{ab}^{cd}(u, v) = \frac{(u - v)\delta_a^c\delta_b^d + i\delta_a^d\delta_b^c}{u - v - i}. \quad (4.1)$$

In the first step the indices $a, b, c, d = 2, \dots, N$ took $N - 1$ values, and in the last step they were all fixed $a, b, c, d = N$.

Since nothing much changed in each step of the nested Bethe ansatz, we can take a step backwards from the first level and consider the so-called *R-matrix*

$$R_{ab}^{cd}(u, v) = \frac{(u - v)\delta_a^c\delta_b^d + i\delta_a^d\delta_b^c}{u - v + i}, \quad (4.2)$$

where $a, b, c, d = 1, \dots, N$. This matrix enjoys the full $SU(N)$ symmetry of the spin chain as well as a couple of features to be discussed below which make it ideally suited for the construction and investigation of quantum integrable models. It differs from the above scattering matrices by an overall prefactor of $(u - v - i)/(u - v + i)$ which will be largely inconsequential but convenient.

R-Matrix Notations. Before we discuss the properties of R-matrices which come to use in the construction of integrable systems we shall introduce some notation for combining R-matrices which is very helpful for working out identities.

The R-matrix is a linear operator acting on the tensor square of the vector space $\mathbb{V} = \mathbb{C}^N$

$$\mathcal{R} : \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}. \quad (4.3)$$

Moreover, it depends on one complex parameter associated to each of the two vector spaces, $\mathcal{R} = \mathcal{R}(u, v)$. Here it makes sense to also allow the point at infinity $u, v = \infty$ as a parameter value. In fact, the R-matrix depends only on the

difference of these parameters $\mathcal{R}(u, v) = \mathcal{R}(u - v)$, but we will hardly make use of this feature.

More concretely, the R-matrix takes the form

$$\mathcal{R}(u, v) = \frac{(u - v)\mathcal{I} + i\mathcal{P}}{u - v + i}, \quad (4.4)$$

where \mathcal{I} and \mathcal{P} denote the identity and permutation operators acting on $\mathbb{V} \otimes \mathbb{V}$. Note that the vector space \mathbb{V} is a representation space of the Lie group $SU(N)$, and the R-matrix is symmetric under the canonical action of $SU(N)$ on the tensor product $\mathbb{V} \otimes \mathbb{V}$.

Introducing a basis $\{E^a\}$ for the vector space \mathbb{V} and a dual basis $\{E_a\}$ for \mathbb{V}^* we can decompose the R-matrix into components \mathcal{R}_{ab}^{cd} ,

$$\mathcal{R} = (E^a \otimes E^b) R_{ab}^{cd} (E_c \otimes E_d). \quad (4.5)$$

In that sense, the R-matrix is actually a tensor of rank 4 with N^4 components (most of which are zero).¹ The components read

$$R_{ab}^{cd}(u, v) = \frac{(u - v)\delta_a^c \delta_b^d + i\delta_a^d \delta_b^c}{u - v + i}. \quad (4.6)$$

They are formulated in terms of Kronecker symbols δ_b^a which are invariant under $SU(N)$ by construction

Let us now introduce an abbreviated symbolic and a graphical notation to deal with operators acting on tensor products of vector spaces \mathbb{V} such as the R-matrix. In order to distinguish the vector spaces within the tensor product, each space receives a label \mathbb{V}_k .

In the symbolic notation, some operator $\mathcal{X}_{k,\dots,m}$ acts linearly on a tensor product of spaces

$$\mathcal{X}_{k,\dots,m} : \mathbb{V}_k \otimes \dots \otimes \mathbb{V}_m \rightarrow \mathbb{V}_k \otimes \dots \otimes \mathbb{V}_m. \quad (4.7)$$

This operator can also act on a tensor product with additional vector spaces, in which case it is assumed to act on the latter as the identity. For example, $\mathcal{X}_{23} := \mathcal{I}_1 \otimes \mathcal{X}_{23}$ when acting on $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$ and $\mathcal{X}_{23} := \mathcal{I}_1 \otimes \mathcal{X}_{23} \otimes \mathcal{I}_4$ when acting on $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3 \otimes \mathbb{V}_4$.

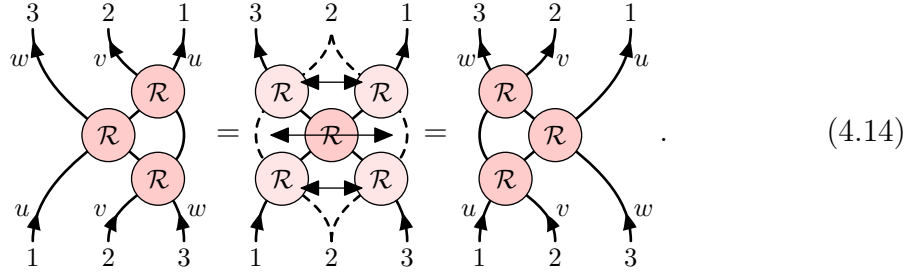
The R-matrix acts on a pair of spaces $\mathbb{V}_k, \mathbb{V}_l$ with associated parameters u_k, u_l . A useful shorthand notation is

$$\mathcal{R}_{k,l} := \mathcal{R}_{k,l}(u_k, u_l) : \mathbb{V}_k \otimes \mathbb{V}_l \rightarrow \mathbb{V}_k \otimes \mathbb{V}_l. \quad (4.8)$$

The short notation is sufficient because the parameters are linked tightly to the spaces.

¹ The term ‘‘matrix’’ refers to the fact that the R-matrix is a linear operator and can thus be written as a matrix. Often, R-matrices are written in $N^2 \times N^2$ matrix notation where the two ingoing and two outgoing indices are (implicitly) combined into a composite index.

The graphical representation of the YBE takes an inspiring form

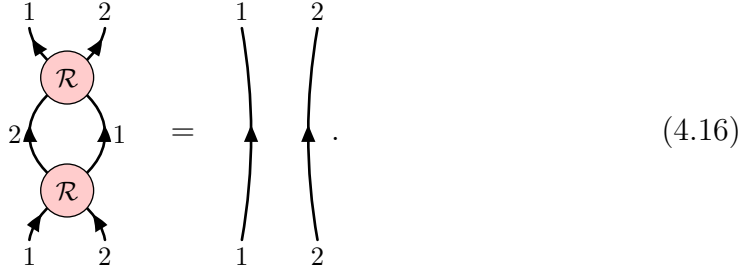


When considering a tangle of lines with the appropriate R-matrices at each intersection, the YBE allows us to shift one strand past an intersection of two other strands. As the figures shows the order of the R-matrices is inverted by such an operation.

Another important property is that \mathcal{R}_{21} is the inverse of \mathcal{R}_{12}

$$\mathcal{R}_{21}\mathcal{R}_{12} = \mathcal{I}. \quad (4.15)$$

The graphical representation for the above property tells us that we can remove a double crossing of two strands and pull them straight



Note that for $\mathcal{R}_{12} = \mathcal{R}_{12}(u, v)$ the operator \mathcal{R}_{21} is defined as

$$\mathcal{R}_{21} := \mathcal{R}_{21}(v, u) = \mathcal{P}_{12}\mathcal{R}_{12}(v, u)\mathcal{P}_{12} = \frac{(u - v)\mathcal{I} - i\mathcal{P}}{u - v - i}. \quad (4.17)$$

The combination of the above two rules leads to an interesting structure

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \quad \mathcal{R}_{12}\mathcal{R}_{21} = \mathcal{I}. \quad (4.18)$$

These are in fact the defining relations of the permutation group, where $\mathcal{R}_{k,l}$ represents a pairwise permutation between two elements labelled k and l . This feature allows to use R-matrices as the pairwise scattering matrix for a factorised scattering problem because for every permutation there is a unique combination of pairwise R-matrices up to identities. For a tangle of lines, the above identities imply that only the permutation between the ingoing and outgoing vector spaces matters. In other words, lines can be deformed at will as long as at every crossing an R-matrix is inserted.

In addition, there are two properties related to special points which will be useful for the construction of integrable systems.

When both parameters are the same, $u = v$, the R-matrix becomes the permutation

$$\mathcal{R}(u, u) = \mathcal{P}, \quad \begin{array}{c} \begin{array}{ccc} 2 & 1 & 2 \\ \swarrow & \nearrow & \uparrow \\ u & u & \\ \nwarrow & \searrow & \downarrow \\ 1 & 2 & 1 \end{array} \\ \mathcal{R} \end{array} = \begin{array}{c} \begin{array}{ccc} 2 & & 1 \\ \uparrow & & \uparrow \\ & & \\ & & \\ & & \\ & & \downarrow \\ & & 1 \end{array} \\ \mathcal{P} \end{array}. \quad (4.19)$$

In the scattering context, this identity implies identical particles.

For the class of rational R-matrices the points $u = \infty$ or $v = \infty$ are also special. Here the R-matrix trivialises to the identity

$$\mathcal{R}(u, \infty) = \mathcal{R}(\infty, v) = \mathcal{I}, \quad (4.20)$$

or graphically

$$\begin{array}{c} \begin{array}{ccc} 2 & 1 & \\ \swarrow & \nearrow & \\ \infty & u & \\ \nwarrow & \searrow & \\ u & & \infty \\ 1 & 2 & \end{array} \\ \mathcal{R} \end{array} = \begin{array}{c} \begin{array}{ccc} 2 & 1 & \\ \swarrow & \nearrow & \\ v & \infty & \\ \nwarrow & \searrow & \\ \infty & v & \\ 1 & 2 & \end{array} \\ \mathcal{R} \end{array} = \begin{array}{c} \begin{array}{ccc} 2 & & 1 \\ \uparrow & & \uparrow \\ & & \\ & & \\ & & \\ & & \downarrow \\ & & 1 \end{array} \\ \mathcal{I} \end{array}. \quad (4.21)$$

In the scattering context, this identity relates a magnon at zero momentum to a symmetry of the system.

R-Matrix Generalisations. Note that the above R-matrix is one of the simplest ones that exist; there are many much more elaborate generalisations. Let us summarise a few of them briefly which share most of the above properties:

- The space \mathbb{V} can be replaced by different representation spaces potentially of a different Lie group.
- The above R-matrix has no parameters beyond those associated to the two vector spaces. Most R-matrices allows for several globally defined deformation parameters. These deformation parameters may alter or spoil the properties associated to the special point $u, v = \infty$.
- Our R-matrix was defined on the tensor square of the space \mathbb{V} . R-matrices can also be defined for tensor products of inequivalent spaces $\mathbb{V}_k, \mathbb{V}_k$. In this case many of the discussed properties only hold when introducing one R-matrix for each pair of admissible spaces. Alternatively, one could consider the direct sum of all admissible vector spaces with a single R-matrix acting on the tensor square.
- We considered the case where every vector space has one associated parameter. Generalisations of this concept involve several (or no) parameters associated to a vector space.
- The R-matrix considered above depends on the difference of its two parameters. Most known examples obey such a difference form. There are, however, notable exceptions where the R-matrix is not of difference form (even after taking into account reparametrisations discussed below).

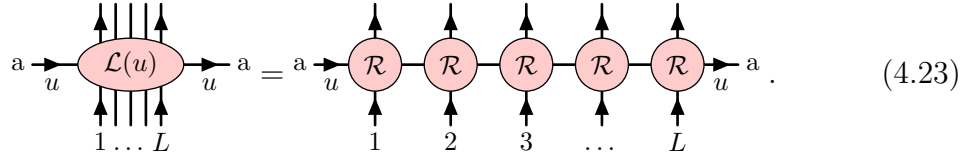
- One can apply a map $u \mapsto f(u)$ to the parameters of \mathcal{R} without spoiling most of the properties discussed above (the location of the special points evidently changes). For example, our S-matrix was initially given in terms of momenta p, q instead of rapidities u, v . The difference property, however, singles out a preferred choice of parameters.
- For most purposes, the overall normalisation of the R-matrix does not matter. For instance, one often considers polynomial R-matrices without the denominator of our R-matrix was removed. Such extra factors modify some of the above relations slightly. For instance, $\mathcal{R}_{21}\mathcal{R}_{12}$ will equal the identity merely up to some overall factor.

Monodromy and Transfer Matrices. In classical field theory, we introduced the Lax connection as a formulation of integrability. For spin chains, however, space is discrete and states are quantum. Therefore, a Lax connection is not applicable; it is replaced by the R-matrix. The R-matrix takes the analogous role of the parallel transport of the Lax connection past one site of the spin chain which is the minimum distance that makes sense for this model.³ Moreover, the R-matrix is a quantum operator rather than a function of phase space.

Next let us now consider the spin chain as a whole. We assume the chain to have closed boundary conditions. Using the above analogy with classical field theory, the monodromy matrix $\mathcal{L}(u)$ for the spin chain (which describes half of the Lax pair) is pieced together from an R-matrix for each site

$$\mathcal{L}_a(u) = \mathcal{R}_{a,L}\mathcal{R}_{a,L-1}\dots\mathcal{R}_{a,2}\mathcal{R}_{a,1}. \quad (4.22)$$

In graphical notation the monodromy matrix reads



$$\begin{array}{c} \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \uparrow \\ \text{a} \rightarrow \text{L}(u) \rightarrow \text{a} \\ \uparrow \uparrow \uparrow \uparrow \uparrow \\ 1 \dots L \end{array} = \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \uparrow \\ \text{a} \rightarrow \text{R} \text{---} \text{R} \text{---} \text{R} \text{---} \text{R} \text{---} \text{R} \rightarrow \text{a} \\ \uparrow \uparrow \uparrow \uparrow \uparrow \\ 1 \quad 2 \quad 3 \quad \dots \quad L \end{array} \end{array} \quad (4.23)$$

The monodromy matrix $\mathcal{L}_a(u)$ is a matrix of operators which act on the Hilbert space of the quantum spin chain. Note that the parameter $u = u_a$ is associated to the auxiliary space \mathbb{V}_a on which the matrix acts (i.e. the space of the classical Lax pair). The parameters v_k associated to the spin sites are fixed to some values. Since we are interested in *homogeneous* spin chain models we choose all parameters to be equal, conveniently $v_k = 0$.⁴

It is now straight-forward to construct charges in involution as the trace of the monodromy matrix

$$\mathcal{F}(u) = \text{tr}_a \mathcal{L}_a(u). \quad (4.24)$$

³The R-matrix serves many purposes in spin chains. In this context it is also known as the Lax matrix, typically with a convenient prefactor and one of the two parameters u, v fixed to a particular value.

⁴An *inhomogeneous* spin chain with $v_k \neq v_l$ is integrable as well and can be treated with minor modifications.

This so-called transfer matrix $\mathcal{F}(u)$ can be written graphically as⁵

The diagram shows three equivalent representations of the transfer matrix $\mathcal{F}(u)$. On the left, a pink oval labeled $\mathcal{F}(u)$ has L vertical lines entering from the bottom and exiting from the top, labeled $1 \dots L$. In the middle, a pink oval labeled $\mathcal{L}(u)$ has L vertical lines passing through it, with a loop labeled u that goes around the oval. On the right, a chain of L pink circles, each with a vertical line passing through it, has a loop labeled u that goes around the entire chain. The equations are: $\mathcal{F}(u) = \mathcal{L}(u) =$ (chain of circles with loop).

Two transfer matrices commute at arbitrary spectral parameters u, v

$$[\mathcal{F}(u), \mathcal{F}(v)] = 0. \quad (4.26)$$

To see this is quite evident in graphical notation where we need to show the following equality

The diagram shows two equivalent configurations of two transfer matrices. On the left, two chains of pink circles are shown. The top chain has a loop labeled v and the bottom chain has a loop labeled u . On the right, the loops are swapped: the top chain has a loop labeled u and the bottom chain has a loop labeled v . The equality is shown by a central equals sign.

We have already learned that we can deform the lines and move them past others lines and intersections. This allows to move the upper loop past the lower loop and thus switch their ordering.

A symbolic proof is also straight-forward, but requires several steps which are not as easy to spot in a long sequences of symbols. The first step is to let the two loops overlap somewhere by inserting an R-matrix and its inverse. The next step consists in pulling the upper loop below the lower loop past all intersections along the spin chain. In a final step the R-matrix and its inverse are removed by pulling the loops apart. These three steps look as follows

The diagram shows three stages of the proof. Stage 1: Two chains of pink circles with loops labeled u and v overlapping. Stage 2: The loops are separated by an R-matrix (a pink circle with two lines crossing). Stage 3: The loops are completely separated, with the u loop now below the v loop. The stages are connected by equals signs.

Local Charges. Next we have to define a suitable Hamiltonian for the system. This should be constructed out of the transfer matrix $\mathcal{F}(u)$ such that it automatically commutes with the transfer matrices $\mathcal{F}(v)$ at arbitrary values v . We shall see that the expansion around the point $u = 0$ (which coincides with the values of the parameters v_k associated to the spin sites) is perfectly suited for this purpose.

Let us therefore expand the R-matrix around the point $u = 0$ with fixed $u_k = 0$. We find

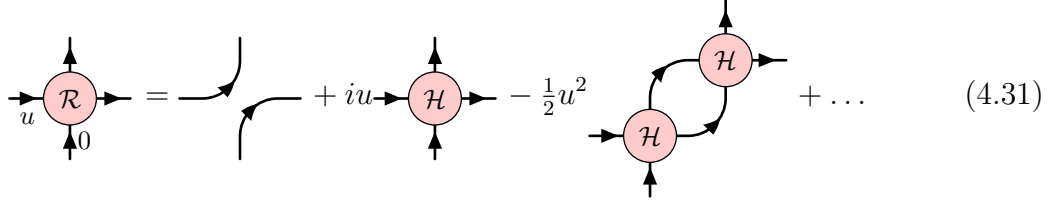
$$\mathcal{R}_{a,k}(u, 0) = \mathcal{P}_{a,k} + iu\mathcal{P}_{a,k}\mathcal{H}_{a,k} - \frac{1}{2}u^2\mathcal{P}_{a,k}\mathcal{H}_{a,k}^2 + \dots \quad (4.29)$$

⁵Since the chain is periodic, the lines should be drawn on the surface of a cylinder. The loop can thus be closed without introducing further crossings.

with the Hamiltonian kernel of the Heisenberg XXX model or its generalisation to the $SU(N)$ fundamental spin chain

$$\mathcal{H}_{k,l} = \mathcal{I}_{k,l} - \mathcal{P}_{k,l}. \quad (4.30)$$

In a graphical notation the expansion can be written as



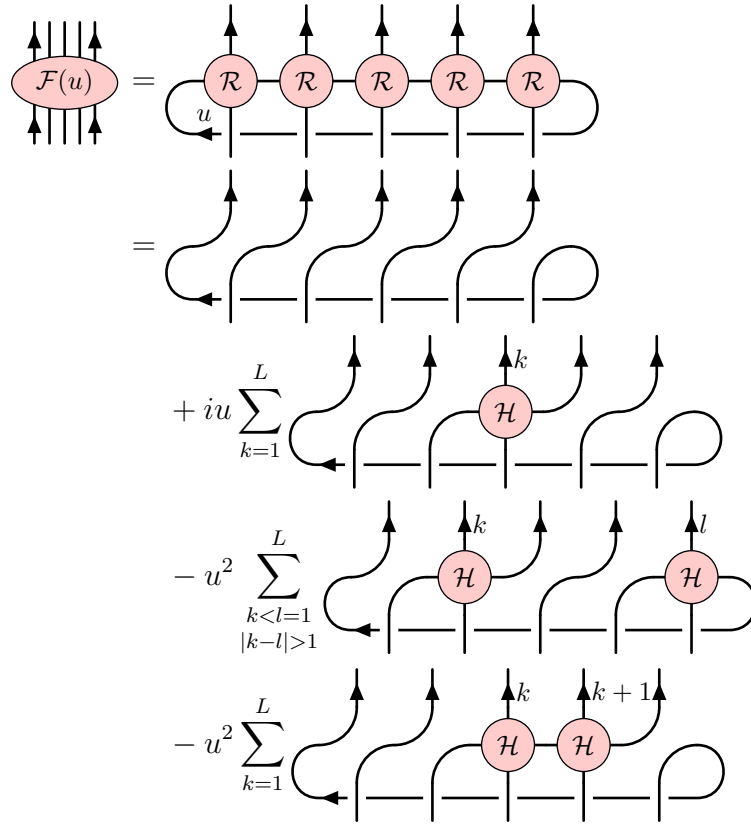
$$\mathcal{R} = \text{crossing} + iu \mathcal{H} - \frac{1}{2}u^2 \text{H-H diagram} + \dots \quad (4.31)$$

with the Hamiltonian kernel taking the form⁶



$$\mathcal{H} = \text{parallel lines} - \text{crossing lines}. \quad (4.32)$$

Now we can expand the transfer matrix $\mathcal{F}(u)$ around the point $u = 0$. Up to second order we find



$$\begin{aligned} \mathcal{F}(u) &= \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \mathcal{R}_4 \mathcal{R}_5 \\ &= \text{curved R-matrices} \\ &\quad + iu \sum_{k=1}^L \text{H kernel} \\ &\quad - u^2 \sum_{\substack{k < l = 1 \\ |k-l| > 1}}^L \text{H-H kernel} \\ &\quad - u^2 \sum_{k=1}^L \text{H-H kernel} \end{aligned}$$

⁶We use a different convention to for the R-matrix and kernels of local operators: The R-matrix is located at the intersection of two crossing lines, whereas the Hamiltonian joins two lines which do not cross. This implies a different ordering for the external legs.

$$\begin{aligned}
& -\frac{1}{2}u^2 \sum_{k=1}^L \text{diagram} \\
& + \dots
\end{aligned} \tag{4.33}$$

Let us discuss the arising terms one at a time:

The leading term describes a cyclic shift \mathcal{U} of the closed chain

$$\text{diagram} = \text{diagram} \tag{4.34}$$

Also the sub-leading terms are cyclic shift operations for most of the legs, so it makes sense to factor out the operator \mathcal{U} .

We denote the term at linear order in u by $-iu\mathcal{U}\mathcal{H}$. The operator \mathcal{H} is in fact the Hamiltonian given by a homogeneous sum of Hamiltonian kernels around the closed chain

$$\text{diagram} = \sum_{k=1}^L \text{diagram} \tag{4.35}$$

The term on the next line contains two insertions of the Hamiltonian kernels at arbitrary non-overlapping positions of the spin chain. All of these terms are generated by the square of the Hamiltonian $-\frac{1}{2}u^2\mathcal{U}\mathcal{H}^2$. However, one has to pay attention to the terms where two kernels overlap: The terms where two kernels are inserted at the same location is covered precisely by the last line. The second but last line describes terms where the insertions are shifted by one site. Those terms arise in \mathcal{H}^2 , but only with half of the coefficient. Conversely, there are further terms in \mathcal{H}^2 where the order of insertions is flipped. We summarise these additional and missing terms in the operator

$$\text{diagram} = \sum_{k=1}^L \text{diagram} \tag{4.36}$$

It is a local operator with kernel that acts on next-to-nearest neighbours

$$\text{diagram} = \frac{i}{2} \text{diagram} - \frac{i}{2} \text{diagram} \tag{4.37}$$

In formulas we can write this kernel as

$$\mathcal{Q}_{3;k,l,m} = \frac{i}{2}[\mathcal{H}_{l,m}, \mathcal{H}_{k,l}]. \quad (4.38)$$

Altogether we find that the expansion is written nicely as an exponential

$$\mathcal{F}(u) = \mathcal{U} \exp(iu\mathcal{H} + iu^2\mathcal{Q}_3 + \dots). \quad (4.39)$$

The operators \mathcal{Q}_r in the exponent have the relevant property of being *local*. In fact, their kernels extend over an extended range of r sites. In particular, the first operator \mathcal{Q}_2 in this tower is the Hamiltonian $\mathcal{Q}_2 = \mathcal{H}$. Commutativity of the transfer matrices $\mathcal{F}(u)$ and $\mathcal{F}(v)$ at arbitrary values u, v leads to the involution property

$$[\mathcal{Q}_r, \mathcal{Q}_s] = 0. \quad (4.40)$$

We have thus constructed a tower of commuting local operators.

Locality of the integrable charges is a crucial feature for at least two reasons:

- For a Hilbert space of dimension N there always exist $N - 1$ commuting independent operators which also commute with a given Hamiltonian: In a basis where the Hamiltonian is diagonal, these are the remaining independent diagonal matrices. Since this construction does not rely on any special properties of the physical system, it can hardly be useful. In order to be useful, quantum integrability must require further properties for the commuting charges such as locality.
- In the magnon scattering picture, local charges act on the magnons individually provided that the latter are sufficiently well separated. For a state of m magnons, m local commuting charges are needed to guarantee that the momenta are individually conserved. Since the local charges do not distinguish the ordering of magnons along the chain, there are $m!$ partial eigenstates with degenerate charge eigenvalues. These are related by the factorised scattering matrix.

Multi-Local Charges. Another point of interest is $u = \infty$. Here the R-matrix has the following expansion

$$\mathcal{R}_{a,k}(u, 0) = \mathcal{I}_{a,k} + iu^{-1}\mathcal{J}_{a,k} - \frac{1}{2}u^{-2}\mathcal{J}_{a,k}^2 + \dots, \quad (4.41)$$

with the operator

$$\mathcal{J}_{a,k} = \mathcal{P}_{a,k} - \mathcal{I}_{a,k}. \quad (4.42)$$

The expansion in terms of figures reads

$$\begin{array}{c} \uparrow \\ \circlearrowleft \mathcal{R} \\ \uparrow \\ \leftarrow u \end{array} = \begin{array}{c} \uparrow \\ | \\ \rightarrow \end{array} + \frac{i}{u} \begin{array}{c} \uparrow \\ \circlearrowleft \mathcal{J} \\ \uparrow \\ \leftarrow \end{array} - \frac{1}{2u^2} \begin{array}{c} \uparrow \\ \circlearrowleft \mathcal{J}^2 \\ \uparrow \\ \leftarrow \end{array} + \dots \quad (4.43)$$

and the operator \mathcal{J} takes the form⁷

$$\text{Diagram of } \mathcal{J} \text{ (circle with four arrows)} = \text{Diagram of a crossing of two lines}. \quad (4.44)$$

We now expand the monodromy matrix $\mathcal{L}_a(u)$ to second order

$$\begin{aligned} \text{Diagram of } \mathcal{L}(u) &= \text{Diagram of } \mathcal{R} \mathcal{R} \mathcal{R} \mathcal{R} \mathcal{R} \\ &= \text{Diagram of a crossing of two lines} \\ &+ \frac{i}{u} \sum_{k=1}^L \text{Diagram of } \mathcal{J} \text{ at site } k \\ &- \frac{1}{u^2} \sum_{k < l=1}^L \text{Diagram of } \mathcal{J} \text{ at sites } k \text{ and } l \\ &- \frac{1}{2u^2} \sum_{k=1}^L \text{Diagram of } \mathcal{J}^2 \text{ at site } k \\ &+ \dots \end{aligned} \quad (4.45)$$

Let us again discuss the terms that arise: The leading term is merely the identity operator.

At first order we find an operator \mathcal{J}_a which is the sum over the insertions of $\mathcal{J}_{a,k}$ at every site k of the spin chain

$$\text{Diagram of } \mathcal{J}_a = \sum_{k=1}^L \text{Diagram of } \mathcal{J} \text{ at site } k. \quad (4.46)$$

It turns out that $\mathcal{J}_{a,k}$ can be viewed as a fundamental representation of $U(N)$ acting on site k of the chain. Consequently, \mathcal{J}_a is the tensor product representation of $U(N)$ on the whole chain.

At the next order we find again two insertions of $\mathcal{J}_{a,k}$. These terms arise from $-\frac{1}{2}u^{-2}\mathcal{J}_a^2$ which lets \mathcal{L}_a take the form of an exponential. As before, the coefficients of the terms differ by a factor of 2, and an equal number of terms with a different

⁷Here we use the same convention as with the R-matrix that \mathcal{J} resides at the intersection of two lines.

ordering is missing. We summarise them in an operator $\widehat{\mathcal{J}}_a$

$$\begin{aligned}
 \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \rightarrow \text{---} \widehat{\mathcal{J}}_a \text{---} \rightarrow \\ \uparrow \uparrow \uparrow \uparrow \end{array} &= \frac{i}{2} \sum_{k < l = 1}^L \begin{array}{c} \uparrow \\ | \\ \text{---} \mathcal{J}^k \text{---} \\ | \\ \uparrow \end{array} \begin{array}{c} \uparrow \\ | \\ \text{---} \mathcal{J}^l \text{---} \\ | \\ \uparrow \end{array} \rightarrow \\
 &- \frac{i}{2} \sum_{k < l = 1}^L \begin{array}{c} \uparrow \\ | \\ \text{---} \mathcal{J}^k \text{---} \\ | \\ \uparrow \end{array} \begin{array}{c} \uparrow \\ | \\ \text{---} \mathcal{J}^l \text{---} \\ | \\ \uparrow \end{array} \rightarrow . \quad (4.47)
 \end{aligned}$$

This operator turns out to be the generator of an extended symmetry to be discussed further below.

Altogether we find for the expansion of the monodromy

$$\mathcal{L}_a(u) = \exp(iu^{-1} \mathcal{J}_a + iu^{-2} \widehat{\mathcal{J}}_a + \dots). \quad (4.48)$$

From the above discussion it should be evident that the expansion of $\mathcal{L}(u)$ around $u = \infty$ yields a tower of *multi-local* charges. These act at several sites of the spin chain at the same time. The form of the expansion is very special, and related to the fact that the R-matrix reduces to the identity matrix at the point $u = \infty$. At generic points u , the monodromy matrix expands into a set of operators which act non-locally on the spin chain without apparent order.

4.2 Other Types of Bethe Ansätze

The Bethe equations describe the spectrum of quantum spin chains, but there are several ways in which they can be derived and formulated. The various approaches lead to different perspectives, which may be particularly helpful in addressing specific kinds of problems. In the following we present the main few approaches.

Algebraic Bethe Ansatz. We can apply the R-matrix formalism to construct eigenstates of the closed spin chain. This method is not only closer to the quantum field theory formalism, but it also largely based in algebra.

Let us first investigate the monodromy matrix for the Heisenberg XXX spin chain with $N = 2$. The monodromy matrix \mathcal{L} is a 2×2 matrix acting on the auxiliary spin site (as well as a big matrix acting on the space of the spin chain)

$$\mathcal{L}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}. \quad (4.49)$$

The components $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are operators acting on the spin chain which obey certain commutation relations. These can be summarised in the so-called *RTT-relations*⁸

$$\mathcal{R}_{ab}(u, v) \mathcal{L}_a(u) \mathcal{L}_b(v) = \mathcal{L}_a(v) \mathcal{L}_b(u) \mathcal{R}_{ab}(u, v). \quad (4.50)$$

⁸The name originates from a notation where the monodromy matrix is assigned the letter T.

In the graphical notation they read

$$\begin{array}{c}
 \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \end{array} \\
 \begin{array}{c} \text{a} \\ \text{b} \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \mathcal{L}(u) \\ \mathcal{L}(v) \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \mathcal{R} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \text{b} \\ \text{a} \end{array} \\
 \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \end{array}
 \end{array} = \begin{array}{c}
 \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \end{array} \\
 \begin{array}{c} \text{a} \\ \text{b} \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \mathcal{R} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \mathcal{L}(v) \\ \mathcal{L}(u) \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \text{b} \\ \text{a} \end{array} \\
 \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \end{array}
 \end{array} . \quad (4.51)$$

They follow straight-forwardly from repeated application of the Yang–Baxter equations.⁹

We know that our R-matrix is invariant under $SU(2)$, in particular, it preserves the number of up and down spins. Consequently, a spin flip in the auxiliary space must be compensated by an opposite spin flip on the spin chain. The components \mathcal{A} and \mathcal{D} do not change the number of up and down spins, whereas \mathcal{B} and \mathcal{C} increase and decrease the number of up spins by one unit, respectively.

Recalling that we treated a spin flip as a magnon particle, the above discussion is reminiscent of the framework of quantum field theory where \mathcal{B} and \mathcal{C} take the roles of creation and annihilation operators, respectively, whereas \mathcal{A} and \mathcal{D} serve as charges. The RTT relations provide the commutation relations which are of the same kind as the commutation relations for (free) particles, but somewhat more involved.

To construct eigenstates we start again with a ferromagnetic vacuum state

$$|0\rangle = |\downarrow\downarrow\dots\downarrow\rangle. \quad (4.52)$$

This state is evidently annihilated by $\mathcal{C}(u)$ for any u . Excited states are generated by acting with several $\mathcal{B}(u)$'s on the vacuum.

$$|u_1, \dots, u_M\rangle = \mathcal{B}(u_1) \dots \mathcal{B}(u_M)|0\rangle. \quad (4.53)$$

This state has M up spins, and therefore it is an M -magnon state. The u_k correspond to the magnon rapidities which are related to the momenta by the relation $u_k = \frac{1}{2} \cot(\frac{1}{2}p_k)$ we used earlier to introduce the rapidity variables. The operator $\mathcal{B}(u_k)$ places a magnon with momentum p_k on top of the existing magnon. This is done precisely in accordance with the rules to assemble multi-magnon states described above. All of this construction is neatly encoded into the R-matrix.

So far we have not specified boundary conditions because the monodromy matrix simply ends at the first and the last sites. For a closed chain the latter should be related as any other pair of adjacent sites. This is achieved by the trace within the transfer matrix $\mathcal{F}(u)$. We are thus interested in its eigenvalues which includes the energy spectrum of the closed chain. Let us therefore act with

$$\mathcal{F}(u) = \mathcal{A}(u) + \mathcal{D}(u) \quad (4.54)$$

⁹The RTT-relations imply that the monodromy matrix $\mathcal{L}(u)$ is an R-matrix as well. This R-matrix is slightly more general than the one we discussed above: It acts on two inequivalent spaces, the auxiliary spin site and the Hilbert space of the spin chain. The spin chain space supplies not just one parameter v , but rather one parameter v_k for each spin site. We merely *decided* to set all these parameters to zero $v_k = 0$.

on a state $|u_1, \dots, u_M\rangle$. This operation can be performed by means of the RTT algebra of the components $\mathcal{A}, \mathcal{B}, \mathcal{D}$. Not too surprisingly, the state is an eigenstate of $\mathcal{F}(u)$ precisely if the magnon rapidities u_k satisfy the closed chain Bethe equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j + i}{u_k - u_j - i} \quad \text{for } k = 1, \dots, M. \quad (4.55)$$

Interestingly, we can now compute the eigenvalue $F(u)$ of $\mathcal{F}(u)$ with full dependence on u

$$F(u) = \prod_{k=1}^M \frac{u - u_k - \frac{i}{2}}{u - u_k + \frac{i}{2}} + \left(\frac{u}{u+i}\right)^L \prod_{k=1}^M \frac{u - u_k + \frac{3i}{2}}{u - u_k + \frac{i}{2}}. \quad (4.56)$$

The two terms roughly correspond to action of the operators $\mathcal{A}(u)$ and $\mathcal{D}(u)$ up to a bunch of extra contributions which cancel between the two terms when the Bethe equations hold.

Above, we have derived a relationship between the expansion of $\mathcal{F}(u)$ at $u = 0$ and some local charges including the Hamiltonian

$$\mathcal{F}(u) = \mathcal{U} \exp(iu\mathcal{H} + iu^2\mathcal{Q}_3 + \dots). \quad (4.57)$$

The same relationship evidently holds for the eigenvalues. We thus find¹⁰

$$\begin{aligned} F(u) &= U \exp(iuE + iu^2Q_3 + \dots), \\ U &= \prod_{k=1}^M \frac{-u_k - \frac{i}{2}}{-u_k + \frac{i}{2}}, \\ E &= \sum_{k=1}^M \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}} \right), \\ Q_3 &= \sum_{k=1}^M \left(\frac{i}{2(u_k + \frac{i}{2})^2} - \frac{i}{2(u_k - \frac{i}{2})^2} \right). \end{aligned} \quad (4.58)$$

A benefit of this so-called *algebraic Bethe ansatz* is that it is readily generalised to bigger symmetry algebras. Let us sketch how to apply to apply the algebraic Bethe ansatz to the $SU(N)$ fundamental spin chain. We first decompose the monodromy matrix as follows

$$\mathcal{L} = \begin{pmatrix} \mathcal{A}^1 & \mathcal{B}^1 & * & * \\ \mathcal{C}^1 & \mathcal{A}^2 & \ddots & * \\ * & \ddots & \ddots & \mathcal{B}^{N-1} \\ * & * & \mathcal{C}^{N-1} & \mathcal{A}^N \end{pmatrix}. \quad (4.59)$$

¹⁰All of the *local* charge eigenvalues originate from the first term only because the second term is suppressed by u^L . At sufficiently large order this term also contributes, but the corresponding charges can hardly be called local because they extend over whole length of the spin chain.

The operators \mathcal{B}^r and \mathcal{C}^r serve as creation and annihilation operators of $N - 1$ kinds. They correspond one-to-one to the various excitations of the nested Bethe ansatz. The operators \mathcal{A}^r on the diagonal leaves the numbers of all kinds of excitations invariant.¹¹ We will not need the other operators explicitly because they can be written as combinations of the above elementary building blocks. A generic state is then written as

$$|u_k^r, u_l^s, \dots\rangle = \mathcal{B}^r(u_k^r)\mathcal{B}^s(u_l^s)\dots|0\rangle. \quad (4.60)$$

where the vacuum is again the state with all spins aligned such that it is annihilated by all \mathcal{C}^r .

Analytic Bethe Ansatz. Let us reconsider the eigenvalue of the transfer matrix from the algebraic Bethe ansatz

$$F(u) = \prod_{k=1}^M \frac{u - u_k - \frac{i}{2}}{u - u_k + \frac{i}{2}} + \left(\frac{u}{u+i}\right)^L \prod_{k=1}^M \frac{u - u_k + \frac{3i}{2}}{u - u_k + \frac{i}{2}}. \quad (4.61)$$

Compare this to the definition of the transfer matrix

$$\mathcal{F}(u) = \text{tr}_a \mathcal{R}_{a,L} \mathcal{R}_{a,L-1} \dots \mathcal{R}_{a,2} \mathcal{R}_{a,1}, \quad \mathcal{R}_{a,k} = \frac{u\mathcal{I}_{a,k} + i\mathcal{P}_{a,k}}{u+i}. \quad (4.62)$$

One immediately observes that $\mathcal{F}(u)$ is a rational function with an L -fold pole at $u = -i$, no other poles and $\mathcal{F}(\infty) = 2\mathcal{I}$. The eigenvalue $F(u)$ has the same properties, but additional apparent poles at $u = u_k - \frac{i}{2}$. How do these observations fit together? Did something go wrong?

Let us therefore investigate the residue of $F(u)$ at the dynamical pole $u = u_k - \frac{i}{2}$

$$\begin{aligned} & F(u_k - \frac{i}{2} + \epsilon) \\ & \sim -\frac{i}{\epsilon} \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j + i}{u_k - u_j} + \frac{i}{\epsilon} \left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}}\right)^L \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j - i}{u_k - u_j} \\ & \sim -\frac{i}{\epsilon} \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j + i}{u_k - u_j} \left(1 - \left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}}\right)^L \prod_{\substack{j=1 \\ j \neq k}}^M \frac{u_k - u_j - i}{u_k - u_j + i}\right). \end{aligned} \quad (4.63)$$

This shows that the residue is zero whenever the Bethe equations are satisfied, and there are no unwanted dynamical poles.

We can use the above observations to formulate the *analytic Bethe ansatz*: Suppose we are given the above form of the transfer matrix eigenvalue function $F(u)$ with unknown Bethe roots u_k . The u_k are then constrained by the requirement that $F(u)$ has no poles other than an L -fold pole at $u = -i$. In other words, $F(u)$ must be an analytic function except at $u = -i$.

¹¹In the context of Lie algebra the above operators correspond to *Chevalley-Serre generators* and *simple roots* of the algebra.

Baxter Equation. Recall the eigenvalue of the transfer matrix and introduce the slightly modified but more symmetric function $T(u) = (u + \frac{i}{2})^L F(u - \frac{i}{2})$

$$T(u) = (u + \frac{i}{2})^L \prod_{k=1}^M \frac{u - u_k - i}{u - u_k} + (u - \frac{i}{2})^L \prod_{k=1}^M \frac{u - u_k + i}{u - u_k}. \quad (4.64)$$

This function is polynomial of degree L with leading term $2u^L$. Furthermore, introduce the so-called Baxter Q-function $Q(u) = \prod_{k=1}^M (u - u_k)$. The above expression takes the form

$$T(u)Q(u) = (u + \frac{i}{2})^L Q(u - i) + (u - \frac{i}{2})^L Q(u + i). \quad (4.65)$$

On the one hand, the equation defines $T(u)$ as a function of $Q(u)$. On the other hand, it takes the form of a difference equation for $Q(u)$ which is known as the *Baxter equation*.

An important insight is: With the further assumption that $T(u)$ and $Q(u)$ are unknown *polynomials*, the Baxter equation becomes equivalent to the Bethe equations! Some comments

- The roots of the polynomial $Q(u)$ are the Bethe roots.
- $T(u)$ describes the transfer matrix for a given set of Bethe roots encoded into $Q(u)$.
- For any given $T(u)$, there are two solutions of the Baxter equation because the difference equation is of second order.
- The difference equation can be viewed as a quantisation of a differential equation describing classical physics.
- The Baxter equation generalises to many other integrable systems. In particular it can be formulated for models where the coordinate Bethe ansatz does not apply, such as the Heisenberg XYZ chain. In the latter example, the functions T and Q are not polynomials but rather elliptic functions with two periodicities on the complex plane.

T-System. We defined the transfer matrix as a trace of a monodromy matrix with an auxiliary space transforming in the fundamental representation of $SU(N)$. The concept of transfer matrices can be generalised easily to auxiliary spaced transforming in higher representations. The higher transfer matrices all commute with each other at arbitrary parameters.

The eigenvalue of the spin-1 transfer matrix for the Heisenberg XXX spin chain reads

$$\begin{aligned} T_1(u) = & (u + i)^L \prod_{k=1}^M \frac{u - u_k - \frac{3i}{2}}{u - u_k + \frac{i}{2}} + (u - i)^L \prod_{k=1}^M \frac{u - u_k + \frac{3i}{2}}{u - u_k - \frac{i}{2}} \\ & + u^L \prod_{k=1}^M \frac{u - u_k - \frac{3i}{2}}{u - u_k + \frac{i}{2}} \frac{u - u_k + \frac{3i}{2}}{u - u_k - \frac{i}{2}}. \end{aligned} \quad (4.66)$$

We have written the eigenvalue as a polynomial in analogy to $T_{1/2}(u) := T(u)$ vs. the original rational function $F(u)$.

These transfer matrices do not necessarily carry additional information, they merely reshuffle the available information. For instance there is a simple relationship between $T_{1/2}$ and T_1

$$T_{1/2}(u + \frac{i}{2})T_{1/2}(u - \frac{i}{2}) = u^L T_1(u) + (u + i)^L (u - i)^L. \quad (4.67)$$

This identity neatly reflects the $SU(2)$ multiplication rule $(\frac{1}{2}) \otimes (\frac{1}{2}) = (1) \oplus (0)$. The relationship can be understood as follows: We first act with two monodromy matrices with fundamental auxiliary sites on the state. To turn them into transfer matrices, we should take a trace on each auxiliary space (l.h.s.). However, we may also project the tensor product to the spin-1 and spin-0 components first (r.h.s.).¹² The shift of the parameters u by $\pm \frac{i}{2}$ is a quantum effect. It is related to the fact that the tensor product only splits up when the parameters differ by i .

A generalisation of the above identity corresponding to $(\frac{1}{2}n) \otimes (\frac{1}{2}) = (\frac{1}{2}(n+1)) \oplus (\frac{1}{2}(n-1))$ reads

$$T_{n/2}(u + \frac{i}{2})T_{1/2}(u - \frac{i}{2}n) = (u - \frac{i}{2}(n-1))^L T_{(n+1)/2}(u) + (u - \frac{i}{2}(n+1))^L T_{(n-1)/2}(u+i). \quad (4.68)$$

This identity allows to recursively construct transfer matrix eigenvalues for representations with arbitrary spin

$$T_{n/2}(u) = \sum_{r=0}^n (u + \frac{i}{2}(n-2r))^L \prod_{k=1}^M \frac{u - u_k + \frac{i}{2}(n+1)}{u - u_k + \frac{i}{2}(n+1-2r)} \frac{u - u_k - \frac{i}{2}(n+1)}{u - u_k + \frac{i}{2}(n-1-2r)}. \quad (4.69)$$

In particular for spin-0 we should set $T_0(u) = u^L$.

There are many similar identities that relate the various transfer matrix eigenvalues. A very useful generalisation is the defining relation of the T -system¹³

$$T_{n/2}(u + \frac{i}{2})T_{n/2}(u - \frac{i}{2}) = T_{(n+1)/2}(u)T_{(n-1)/2}(u) + T_{n/2}^+(u)T_{n/2}^-(u). \quad (4.70)$$

This equation has the form of a difference equation reminiscent of the Hirota equation. It is defined on a lattice of points given by three rows T_s^+ , T_s and T_s^- .

$$\begin{array}{cccccc}
 (T_0^+) & (T_{1/2}^+) & (T_1^+) & (T_{3/2}^+) & (T_2^+) & \\
 (T_0) & (T_{1/2}) & (T_1) & (T_{3/2}) & (T_2) & \dots \\
 (T_0^-) & (T_{1/2}^-) & (T_1^-) & (T_{3/2}^-) & (T_2^-) & \\
 \end{array} \quad (4.71)$$

¹²Note that the product of two transfer matrices with rearranged connections due to the projections essentially winds twice around the closed chain. Therefore T_1 can be viewed as the analog of $F_2 = \text{tr} L^2$ which takes two loops around the chain before closing (with suitable modifications for the quantum case).

¹³A reformulation of the T-system is the so-called Y -system. It uses a different set of variable functions Y to eliminate some unphysical degrees of freedom, but has a very similar form otherwise.

In our case the top and bottom rows as well as the first site of the middle row are given by fixed boundary values

$$T_{n/2}^{\pm}(u) = \left(u \pm \frac{i}{2}(n+1)\right)^L, \quad T_0(u) = u^L. \quad (4.72)$$

The middle row contains the dynamical information on the system. Requiring that all the $T_s(u)$ are polynomials is equivalent to the Bethe equation and determines the spectrum of the closed Heisenberg XXX spin chain.

The benefit of the T-system equation is that it generalises to much more complicated systems such as integrable quantum field theories and the so-called thermodynamic Bethe ansatz. To that end one has to set up a suitable lattice of functions¹⁴ and specify appropriate boundary conditions. The drawback of this approach is that the T-system consists of infinitely many functions to be solved simultaneously.

¹⁴The lattice is closely related to the symmetry algebra: The vertical direction corresponds to the Dynkin diagram; in our case there is just a single row (after removing the boundaries) corresponding to the single node of the Dynkin diagram for $SU(2)$. The horizontal direction corresponds to a reduced n -fold symmetric product of the fundamental representation corresponding to one node of the Dynkin diagram; in our case this is the representation with spin $(n/2)$.

5 AdS/CFT Integrability

Topics in this chapter:

- CFT, local operators, anomalous dimensions,
- $SU(2)$ sector, spin chain picture, Hamiltonian
- $SL(2) = SU(1, 1)$ sector, Bethe equations, $PSU(2, 2|4)$ spin chain
- higher-loop corrections, long-range interactions, long-range Bethe equations
- wrapping terms, Lüscher Corrections, very high loop anomalous dimensions, multiple zeta values
- mirror theory, TBA, T/Y-system, NLIE, quantum curve thermodynamic limit
- three-point functions, partial scalar products, determinant formulas
- scattering amplitudes, null polygonal Wilson loops, worldsheet area, T-duality, weak-weak duality, dual conformal symmetry, Yangian symmetry, problem of divergences

For further details, please refer to reviews on this subject.