Introduction to String Theory

Lecture Notes

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0 Overview

String theory is an attempt to quantise gravity and unite it with the other fundamental forces of nature. It combines many interesting topics of (quantum) field theory in two and higher dimensions. This course gives an introduction to the basics of string theory.

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Indicated are the approximate number of 45-minute lectures. Altogether, the course consists of 39 lectures.

0.2 References

There are many text books and lecture notes on string theory. Here is a selection of well-known ones:

- classic: M. Green, J.H. Schwarz and E. Witten, "Superstring Theory" (2 volumes), Cambridge University Press (1988)
- alternative: D. Lüst, S. Theisen, "Lectures on String Theory", Springer (1989).
- new edition: R. Blumenhagen, D. Lüst, S. Theisen, "Basic Concepts of String Theory", Springer (2012).
- standard: J. Polchinski, "String Theory" (2 volumes), Cambridge University Press (1998)
- basic: B. Zwiebach, "A First Course in String Theory", Cambridge University Press (2004/2009)

- recent: K. Becker, M. Becker, J.H. Schwarz, “String Theory and M-Theory: A Modern Introduction”, Cambridge University Press (2007)
- online: D. Tong, “String Theory”, lecture notes, `http://arxiv.org/abs/0908.0333`
- ...

1 Introduction

1.1 Definition

String theory describes the mechanics of one-dimensional extended objects in an ambient space.


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Some features:

- Strings have tension:


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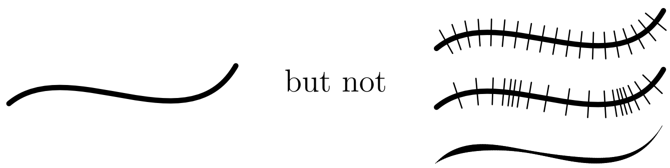
- Strings have no inner structure:



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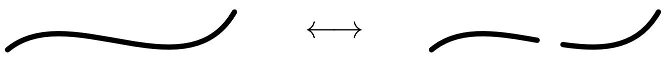
- Several pieces of string can interact:



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- Strings can be classical or quantum:

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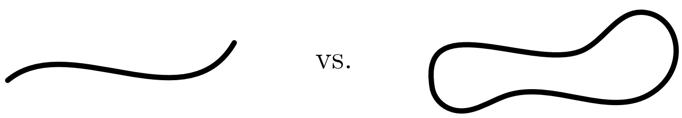
- Strings can be open or closed:



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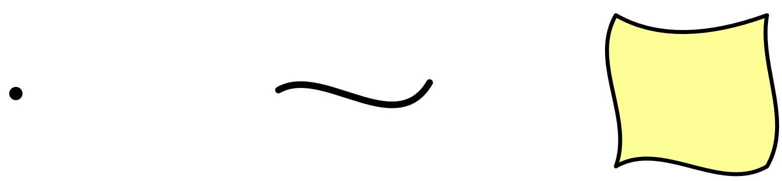
1.2 Motivation

Why study strings?

Extended Objects. We know a lot about the mechanics of point particles. It is natural to study strings next. Or even higher-dimensional extended objects like membranes...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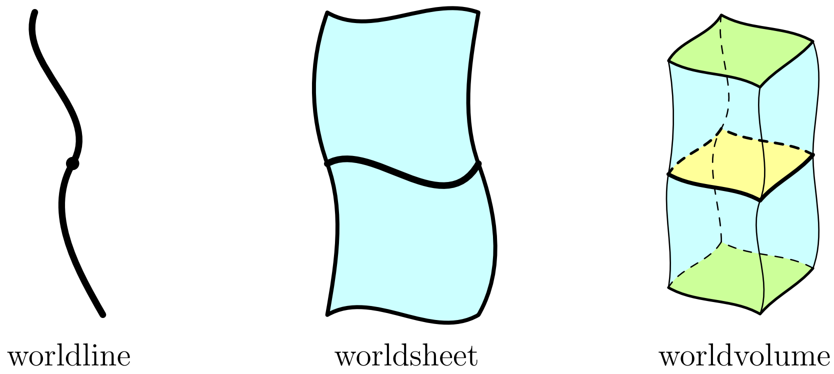
particle string membrane

These objects are snapshots at fixed time t. Introduce the worldvolume as the volume of spacetime occupied by the object:



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The worldsheet of a string is two-dimensional. In fact, there is a great similarity between strings and static soap films.

Quantum Gravity. String theory offers a solution to the problem of quantum gravity (QG). Let us try to sketch the problem of quantum gravity with as little reference to quantum field theory (QFT) as possible.

There are two established classical[1] gravity theories:

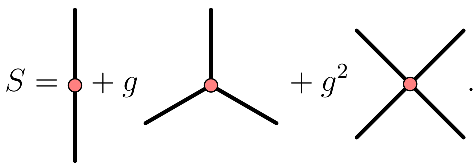
- Newtonian Gravity (non-relativistic)
- General Relativity (GR, relativistic, geometry of spacetime)

We know that nature is quantum mechanical, therefore gravity must also be quantum for consistency with the other fundamental forces. In practice, the effects of QG hardly play a role except for considerations of the early universe and for black hole radiation.

Field quantisation introduces quanta (particles):

- electromagnetism: photon
- strong nuclear forces: gluons
- gravity: graviton
- matter fields: electrons, quarks, neutrinos, …

These particles interact through vertices (Feynman rules) which can be composed to more complex interaction processes (Feynman graphs). The Standard Model (SM) of particle physics has relatively simple set of rules (qualitatively)



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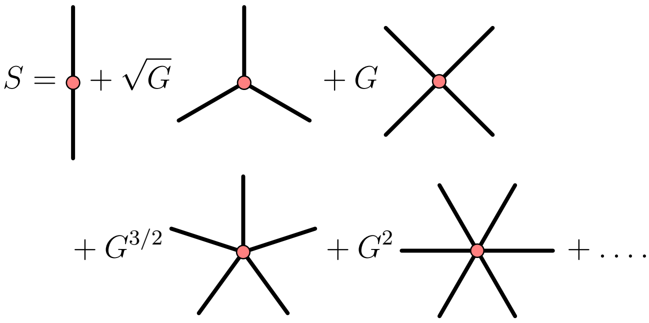
Here ϵ represents a coupling constant.

[1]Here and in the following the term "classical" will refer to the absence of quantum effects. Classical theories can be either non-relativistic or relativistic.

Conversely, Einstein gravity has infinitely many vertices which are governed by Newton's constant G



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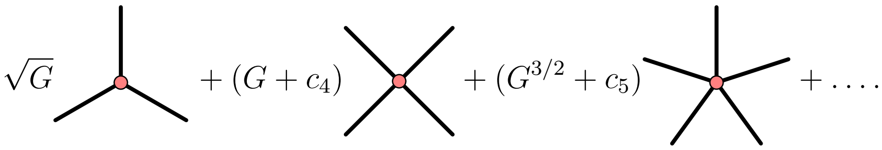
In fact, we can introduce additional couplings c_k:



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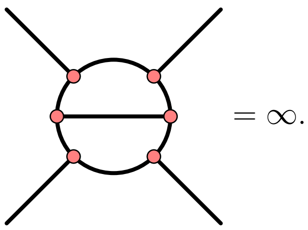
This is perfectly consistent with the assumptions of GR, except that the additional terms introduce higher-derivative corrections to the Einstein equations. Classically we do not need the c_k, but in QFT we do.[2] The point is that loops in Feynman graphs generate divergences, e.g.:



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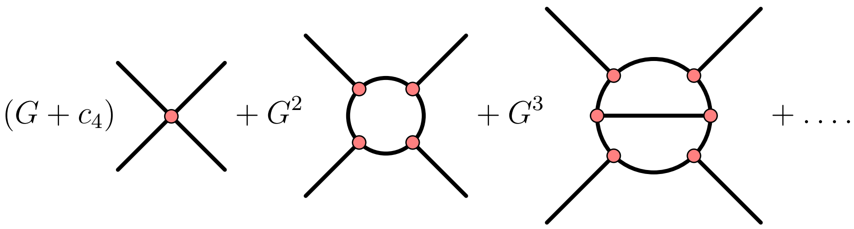
In QFT we have to sum up all competing processes, e.g.:[3]



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In this sum, we can absorb the divergence into a redefinition of the (new) coupling constant $c_4 = -G^3\infty + c_{4,\mathrm{ren}}$. This process is called renormalisation.

All is well now, the divergences are gone, but there is no good way to set the renormalised $c_{4,\mathrm{ren}}$ to zero (or any other distinguished value). Unfortunately,

[2] A general principle of QFT is that we need to include all permissible interaction terms which are not excluded by some principle, typically symmetries or a power counting scheme.

[3] A curious fact is that quantum gravity does not produce a divergence in the one-loop graph (G^2 term).

cancellation of all divergences requires infinitely many c_k's.[4] The quantisation of Einstein gravity introduces infinitely many adjustable parameters. All parameters have to be known (measured) in order to have a predictive description of nature. This renders the theory non-predictive! The only good prediction is at sufficiently low energies much below the Planck scale: There the theory is approximated well by GR with only G as the coupling constant.

What does string theory have to do with it?

Quantum string theory turns out to contain particles which gravitons in many ways. Moreover, string theory does not generate divergences; it is a finite theory! Finally, string theory has just two fundamental coupling constants.

Is all well now!? Almost, many more couplings may be hiding in the description of the vacuum state which is relevant when actual physics is to be addressed.

Unification. String theory provides a unified description for all kinds of fundamental forces of nature.

Electromagnetic and weak forces combine into electroweak forces at sufficiently high energies (around $10^2\,\mathrm{GeV}$). These may also combine with the strong nuclear forces (quantum chromodynamics, QCD) into a so-called Grand Unified Theory (GUT)? There are some hints:

- Charges of fermions appear to suggest larger symmetry group than the one of QCD and the electroweak theory:

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \longleftarrow \mathrm{SU}(5), \mathrm{SO}(10)? \tag{1.14}$$

- Estimated GUT scale $10^{15}\,\mathrm{GeV}$ is somewhere close to Planck scale $10^{19}\,\mathrm{GeV}$. This may suggest a unification of all forces.
- Wouldn't it be nice?

String theory describes gauge theories as well as gravity. In particular, sequence of groups SU(5), SO(10), ... appears.

Does string theory describe nature? So far no convincing derivation. Best option: String theory describes the Standard Model among many (!) other stringy "natures".

String/Gauge Duality. There are intricate relations between string and gauge theories.[5]

- Some effects within gauge theory such as gluon flux tubes in QCD have a stringy nature.
- Some particular gauge theories *are* in fact equivalent to string theories.

String theory can be viewed as an aspect of gauge theory.

[4] In the standard model there are only finitely many coupling constants which require renormalisation. The standard model is said to be renormalisable while general relativity is non-renormalisable.

[5] Gauge theories serve as the description of forces in the standard model.

Treasure Chest. String theory yields many interesting, novel, exceptional structures, results, insights in physics and mathematics. Just to name a few: supersymmetry, higher dimensions, p-branes, dualities, topological insights.

Many Unsolved Problems. Despite 40 years of research there are many unsolved questions:

- How to match with nature?
- How to find direct/indirect evidence?[6]
- What is String Theory?
- How to quantise gravity (otherwise)?

1.3 Some Conventions

Special Relativity.

- Minkowski spacetime $\mathbb{R}^{3,1}$.
- 4-vectors $x^\mu = (t, \vec{x})$; time $t = x^0$.
- indices $\mu, \nu, \ldots = 0, 1, 2, 3$.
- summation convention: $x^\mu y_\mu := \sum_{\mu=0}^{3} x^\mu y_\mu$.
- metric to lower and raise indices: $x_\mu := \eta_{\mu\nu} x^\nu$.
- metric signature: $\eta_{\mu\nu} = \mathrm{diag}(-{+}{+}{+})$, i.e. $p^2 = -m^2$!
- products $x \cdot y := \eta_{\mu\nu} x^\mu y^\nu$; squares $x^2 := x \cdot x$.
- Poincaré symmetry: rotations, Lorentz boosts, spatial and temporal translations.
- generalises to D spacetime dimensions: $\mathbb{R}^{D-1,1}$.

Constants of Nature.

- For simplicity we will set $c = \hbar = e = 1$: All units are then expressed in powers of kg $\sim$ m^{-1}. One can always reconstruct the dependence on $c, \hbar, e$ through dimensional analysis of the expected units.
- Newton's constant G.

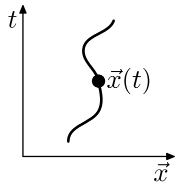
[6]Supersymmetry would be a useful indication if it had been found, but it need to be visible at low energies.

2 Relativistic Point Particle

Let us start slowly with something else: a relativistic particle. Here we will encounter several issues of string theory, but in a more familiar setting. There are many equivalent formulations, we will discuss several.

2.1 Non-Relativistic Actions

We even take another step back and consider a free non-relativistic point particle $\vec{x}(t)$.



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The well-known Lagrange function L and action S read

$$L(\vec{x},\dot{\vec{x}},t)=\tfrac{1}{2}m\dot{\vec{x}}^{\,2},\qquad S[\vec{x}]=\int dt\,L\big(\vec{x}(t),\dot{\vec{x}}(t),t\big). \tag{2.2}$$

The resulting equations of motion (e.o.m.) are just $\ddot{\vec{x}}(t)=0$. The momentum and energy follow from the Hamiltonian formulation

$$\vec{p}=\frac{\partial L}{\partial\dot{\vec{x}}}=m\dot{\vec{x}},\qquad E=H=\frac{\vec{p}^{\,2}}{2m}\,. \tag{2.3}$$

Now promote the above to a relativistic particle

$$L=-mc\sqrt{c^2-\dot{\vec{x}}^{\,2}}=-mc^2+\tfrac{1}{2}m\dot{\vec{x}}^{\,2}+\tfrac{1}{8}mc^{-2}\dot{\vec{x}}^{\,4}+\ldots. \tag{2.4}$$

The expression in the square root is a proper relativistic combination. Its expansion has an irrelevant constant term, the well-known non-relativistic term and relativistic correction terms. Now derive the e.o.m.

$$(c^2-\dot{\vec{x}}^2)\ddot{\vec{x}}+(\dot{\vec{x}}\cdot\ddot{\vec{x}})\dot{\vec{x}}=0. \tag{2.5}$$

The vector nature of the equation implies collinearity of the first and second derivative, $\ddot{\vec{x}}=\alpha\dot{\vec{x}}$. Substitute to obtain the equation $\alpha c^2\dot{\vec{x}}=0$ whose least restrictive solution is $\alpha=0$ which implies $\ddot{\vec{x}}=0$ as in the non-relativistic case. Momentum and energy read

$$\vec{p}=\frac{mc\dot{\vec{x}}}{\sqrt{c^2-\dot{\vec{x}}^{\,2}}}\,,\qquad E=c\sqrt{m^2c^2+\vec{p}^{\,2}}\,. \tag{2.6}$$

This is fine, but not manifestly relativistic: non-relativistic formulation of a relativistic particle. We want a manifestly relativistic formulation using 4-vectors $X^\mu=(ct,\vec{x})$ and $P_\mu=(E/c,\vec{p})$. Let us set $c=1$ for convenience from now on.

- The momentum P_μ is already a good 4-vector:

$$P^2 = -E^2 + \vec{p}^2 = -m^2. \tag{2.7}$$

  The mass shell condition $P^2 = -m^2$ is manifestly relativistic, but $\vec{p}$ and E have a rather distinct role/origin in the Hamiltonian framework.
- The position $X^m(t) = (t, \vec{x}(t))$ and the action $S[\vec{x}]$ make explicit reference to time t (which is defined in a particular Lorentz frame)

$$S = -\int dt\, m\sqrt{-\left(\frac{dX(t)}{dt}\right)^2}. \tag{2.8}$$

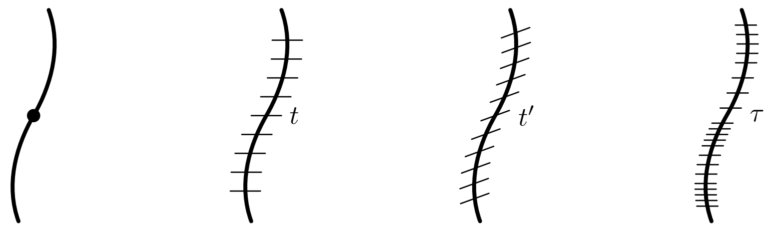
- Note that the Hamiltonian framework makes reference to a slicing of time, hence it distinguishes between space and time.

2.2 Worldline Action

The above action measures Lorentz-invariant proper time s of the particle's path $X^\mu(t)$ in spacetime (worldline)

$$S = -m\int ds, \quad \text{where } ds^2 = -dX^2. \tag{2.9}$$

The proper time depends on the location of the worldline only, but not on a particular Lorentz frame (definition of t) or parametrisation of the worldline (through t).



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Let us assume an arbitrary parametrisation $X^\mu(\tau)$ of the worldline through some curve parameter τ. The proper time action reads (now a dot denotes $d/d\tau$)

$$S = -\int d\tau\, m\sqrt{-\left(\frac{dX(\tau)}{d\tau}\right)^2} = -\int d\tau\, m\sqrt{-\dot{X}^2} \tag{2.11}$$

and is a manifestly relativistic formulation. However, instead of having 3 undetermined functions $\vec{x}(t)$, there are now 4 undetermined functions $X^\mu(\tau)$ and a new function $t(\tau)$.

The equations of motion for the new action read

$$\dot{X}^2\ddot{X}^\mu = (\dot{X}\cdot\ddot{X})\dot{X}^\mu \tag{2.12}$$

and imply collinearity $\ddot{X}^\mu = c\dot{X}^\mu$ for all τ with variable $c(\tau)$. Therefore the resulting worldline is straight.

As a next step, let us derive momenta as derivatives of $L = -m\sqrt{-\dot{X}^2}$ w.r.t. $\dot{X}^\mu$:

$$P_\mu = \frac{m\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\,. \tag{2.13}$$

The above expression immediately squares to $-m^2$; thus the mass shell condition $P^2 = -m^2$ is obeyed.

There are two features special to the above description: while we have only three independent P_μ, there are four independent X^μ. Furthermore, the naive Hamiltonian is strictly zero: $H = 0$. These properties are signals of constraints and gauge invariance:

- Reparametrising $\tau' = f(\tau)$ has no effect on physics.
- Redundancy of description: worldline coordinate τ.
- One linear dependency among the e.o.m. for X^μ.
- Gauge invariance effectively removes one X^μ, e.g. time $t(\tau)$.
- Situation inconvenient for Hamiltonian framework/QM.
- Usually it is better to fix a gauge. There are many choices, so one can pick a convenient one.

In summary, the above action is a fully relativistic formulation, which suffers from the complication of gauge invariance. However, gauge invariance is often considered a virtue: Symmetry! Simultaneously, the above worldline action has two further drawbacks:

- it is non-polynomial; thus inconvenient for quantisation.
- does not work for massless particles $m = 0$.

2.3 Polynomial Action

In order to address the disadvantages from the formulation in the last subsection, let us consider an equivalent action with an auxiliary variable $e(\tau)$

$$S = \int d\tau \left(\tfrac{1}{2}e^{-1}\dot{X}^2 - \tfrac{1}{2}em^2\right). \tag{2.14}$$

The resulting equations of motion read

$$m^2e^2 + \dot{X}^2 = 0, \qquad e\ddot{X}^\mu - \dot{e}\dot{X}^\mu = 0, \tag{2.15}$$

and yield the same old equation for X^μ after combination. The momentum conjugate to X^μ reads $P_\mu = e^{-1}\dot{X}_\mu$; hence the equation of motion for e reduces to $P^2 = -m^2$. The momentum conjugate to e vanishes, signalling a constraint.

While in the previous formulation the massless case did not work, $m = 0$ is a valid choice at every step of the above derivation. It will lead to a constant momentum $P^\mu = e^{-1}\dot{X}^\mu$ as well as $P^2 = 0$. In the massless case, the field e is not fixed by the e.o.m.: there is a remaining gauge freedom.

The einbein e is a dynamical variable. Curiously, the e.o.m. picks out the metric induced by the ambient space. Upon substituting the solution for e, one recovers the above worldline action.

The field e has a nice geometrical interpretation: the *einbein* specifies a metric $g_{\tau\tau} = -e^2$ on the worldline. All terms in the action are combined in a way as to render the action invariant under a change of worldline coordinates (e transforms according to $e' = e\, d\tau'/d\tau$).

In terms of the metric $g_{\tau\tau}$, the above action reads:

$$S = -\tfrac{1}{2} \int d\tau \sqrt{-g_{\tau\tau}} \left(g^{\tau\tau} \dot{X}^2 + m^2 \right) \tag{2.16}$$

where $g^{\tau\tau} = (g_{\tau\tau})^{-1} = -1/e^2$.

2.4 Various Gauges

There is freedom to either fix one of the coordinates $X^\mu(\tau)$ or the auxiliary field e at will. Here are a couple of (more or less) useful choices:

- *Temporal Gauge.* $t(\tau) = \tau$ or $t(\tau) = \alpha\tau$.
  Reduces to non-relativistic treatment of the beginning of the section.
- *Spatial Gauge.* $z(\tau) = \alpha\tau$.
  Works locally except at turning points of $z(\tau)$.
- *Light Cone Gauge.* $x^+(\tau) := t(\tau) + z(\tau) = \alpha\tau$.
  Useful in some cases; prominent in string theory.
- *Proper Time Gauge.* $ds = d\tau$.
  Fixes $t(\tau)$ through the integral

$$t(\tau) = \int^\tau d\tau' \sqrt{1 + \dot{\vec{x}}(\tau')^2}\,. \tag{2.17}$$

  Action becomes trivial $S = -\int d\tau$; dynamics governed by constraint.
- *Constant Einbein.* $\dot{e} = 0$.
  In the polynomial formulation, gauge fixing may involve e. A customary gauge choice is a constant e. In that case, the e.o.m. reduces to

$$\ddot{X} = 0. \tag{2.18}$$

  If the dynamical variable e is gauge fixed to be constant, one must still remember its equation of motion

$$\dot{X}^2 + m^2 e^2 = 0, \tag{2.19}$$

  which turns into a constraint.

2.5 Quantisation

Quantisation can be done in several different ways, depending on the choice of classical formulation. Let us pick the polynomial action discussed in Sec. 2.3. As it is convenient to fix a gauge for the Hamiltonian formulation, we will choose the einbein e to be constant. Momenta P associated to X and the resulting Hamiltonian read:

$$P = e^{-1}\dot{X}, \qquad H = \tfrac{1}{2}e\bigl(P^2 + m^2\bigr). \tag{2.20}$$

Conventionally, a state $|\Psi\rangle$ is given by a wave function of position variables and time

$$|\Psi\rangle = \int d^4X\,\Psi(X,\tau)\,|X\rangle. \tag{2.21}$$

Here, it will be slightly more convenient to immediately Fourier transform to momentum space $|X\rangle \simeq \int d^4P\,e^{iP\cdot X}\,|P\rangle$, resulting in states and wave functions

$$|\Psi\rangle = \int d^4P\,\Psi(P,\tau)\,|P\rangle, \qquad \Psi(P,\tau) = \int d^4X\,e^{iP\cdot X}\Psi(X,\tau). \tag{2.22}$$

respectively. The Schrödinger equation

$$i\dot{\Psi} = H\Psi = \tfrac{i}{2}e\bigl(P^2 + m^2\bigr)\Psi \tag{2.23}$$

is obviously solved by

$$\Psi(P,\tau) = \exp\bigl(-\tfrac{i}{2}e(P^2 + m^2)\tau\bigr)\Phi(P). \tag{2.24}$$

However, the system is constrained. Whenever the constraint $P^2 + m^2 = 0$ is not satisfied, the wave function must vanish:

$$(P^2 + m^2)\Psi(P,\tau) = 0. \tag{2.25}$$

Therefore physical states $\Psi(P,\tau) = \Phi(P)$ are independent of τ, which makes perfect sense: the worldline coordinate τ can be reparametrised and is unphysical. Correspondingly, the Schrödinger equation governing the evolution of τ evolution is replaced by the constraint $P^2 + m^2 = 0$ (which in turn governs the t-evolution).

Upon Fourier-transforming the wave function back to position space $\Phi(X)$, the constraint becomes the Klein–Gordon equation for a spin-0 field

$$(-\partial^2 + m^2)\Phi(X) = 0. \tag{2.26}$$

2.6 Interactions

While the motion of a free particle is easy, one would like to eventually include interactions. Let us sketch how to add interactions with external potentials and with other particles:

Electrical and Gravitational Fields. Coupling the free relativistic particle to electrical and gravitational fields takes a very geometric form

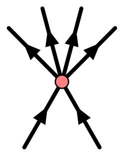
$$S = \int d\tau\,\Bigl(\tfrac{1}{2}e^{-1}g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu - \tfrac{1}{2}em^2 + A_\mu(X)\dot{X}^\mu\Bigr)\,, \tag{2.27}$$

where A_μ is potential for the electromagnetic field and $F_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu$ the corresponding field strength. Likewise $g_{\mu\nu}$ is the gravitational potential, which takes the form of the metric of a curved spacetime.

Here, A_μ and $g_{\mu\nu}$ are assumed to be fixed external fields, that is, they are unaffected by the presence of the particle, but influence its motion. Note however, that those fields are to be evaluated at the dynamical position $X^\mu(\tau)$.

In quantum mechanics, one usually assumes weak interactions, which allows to work with free quantum fields formally. Interactions are then introduced in a perturbative fashion. Whenever a free particle enters a potential field, it scatters off of it. Hereby the dominant contribution originates in single scattering; multiple interactions are suppressed. Only in rare instances, potentials can be handled exactly.

Interactions Among Particles. Local interactions occur, if several particles meet at some spacetime point and split up, potentially into a different number of particles. In the worldline formulation, this type of processes is taken care of by introducing vertices where several particle worldlines meet:



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However, this is not the standard treatment of particle interactions. Usually an interaction of n fields Φ corresponds to a term Φ^n in QFT action. While our method is not very convenient, it works as well. It mimics the Feynman rules and is the standard procedure for string theory.

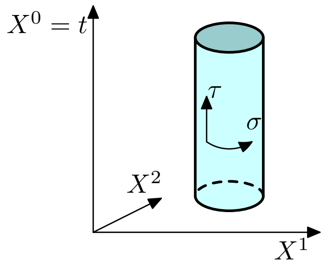
2.7 Conclusions

- We have seen many different formulations of the same physical system and had to deal with gauge invariance and constraints. Depending on the description, there were different numbers of degrees of freedom (d.o.f.), but the number of solutions (modulo gauge) remained the same always.
- We reviewed the quantisation of the free relativistic particle.
- Interactions and couplings to external potentials have been discussed.
- The description here was chosen in light of the analogous treatment in string theory later on. While working well, it was not always the most convenient one.

3 Classical Bosonic String

String theory describes a one-dimensional extended object without inner structure. The main ingredients for its mathematical formulation are

- worldsheet coordinates: $\xi^\alpha = (\tau, \sigma)$, time τ, space σ,
- embedding coordinates: $X^\mu(\xi)$,
- D-dimensional embedding: Minkowski space, metric $\eta_{\mu\nu}$.



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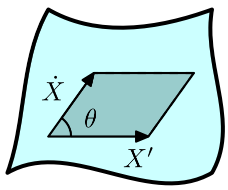
3.1 Nambu–Goto Action

The Nambu–Goto action is the string generalisation of the worldline action described in the last chapter. In order to promote the point particle to a string, the following correspondences are used:

- worldline $\longrightarrow$ worldsheet.
- action = proper time $\simeq$ "length" $\longrightarrow$ "area".

Area and Action. Let us calculate an infinitesimal area element in terms of the embedding coordinates X^μ. After a Wick rotation $t = iw$ the area element dA of 2D euclidean surface reads:

$$\begin{aligned} dA &= d\tau\, d\sigma\, |X'|\, |\dot{X}|\, |\sin\theta| \\ &= d\tau\, d\sigma \sqrt{X'^2\, \dot{X}^2\, \sin^2\theta} \\ &= d\tau\, d\sigma \sqrt{X'^2\, \dot{X}^2 - (X' \cdot \dot{X})^2} \\ &= d^2\xi \sqrt{\det \gamma}\,, \end{aligned} \qquad (3.2)$$



where $\gamma_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ is the induced worldsheet metric (the pull back of the spacetime metric).

Employing another Wick rotation in order to return to a worldsheet with Minkowski signature leads to

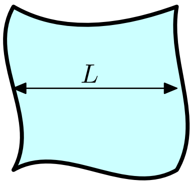
$$S = -\frac{1}{2\pi\kappa^2} A = -\frac{1}{2\pi\kappa^2} \int d^2\xi \sqrt{-\det\gamma}\,, \qquad (3.3)$$

which is the Nambu–Goto action.

The action exhibits the following symmetries:

- *Lorentz symmetry:* because the action is built from scalar products of Lorentz vectors X,
- *Poincaré symmetry:* as there is no explicit dependence on X, only through derivatives ∂X,
- *worldsheet diffeomorphisms:* since the density $d^2\xi\,\sqrt{-\det\gamma}$ is invariant under reparametrisations $\xi \mapsto \xi'(\xi)$.

Tension. What is the meaning of κ? The parameter κ is related to the fundamental string length scale, which will play a role in the quantum string later on.



(3.4)

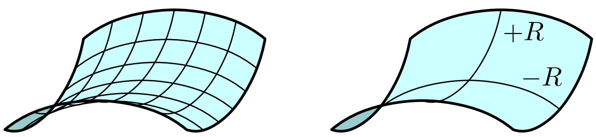
Consider a time slice of the action (and thus a slice of the worldsheet). A slice of length L leads to a potential $U \sim L/\kappa^2$. The constant force resulting from the potential is the string tension: $T = U' = 1/2\pi\kappa^2$.[1]

Equations of Motion. The equations of motion resulting from the Nambu–Goto action can be obtained using the variation of the determinant $\delta\det\gamma = \det\gamma\,\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}$

$$\partial_\alpha\big(\sqrt{-\det\gamma}\,\gamma^{\alpha\beta}\partial_\beta X^\mu\big) = 0. \tag{3.5}$$

As the metric γ contains the embedding X, those equations are highly non-linear and thus difficult to deal with.

What do those equations imply geometrically (in euclidean signature)? The stationary action demands a surface with minimal area, i.e. a static soap film. Thus, the mean curvature is zero everywhere which implies that every point of the surface is a saddle point.



(3.6)

3.2 Polyakov Action

How to remove the complications from the non-linear equations of motion? In the same way as for the point particle, there exists a polynomial action with an additional dynamical worldsheet metric $g_{\alpha\beta}$:

$$S = -\frac{1}{2\pi\kappa^2}\int d^2\xi\sqrt{-\det g}\,\tfrac{1}{2}g^{\alpha\beta}(\partial_\alpha X)\cdot(\partial_\beta X). \tag{3.7}$$

[1]Thus, a string is not a spring or a rubber band because they do not exert a constant force.

The e.o.m. for the field X is the same as above while the e.o.m. for the worldsheet metric g reads:

$$(\partial_\alpha X)\cdot(\partial_\beta X) = \tfrac{1}{2} g_{\alpha\beta}\, g^{\gamma\delta}(\partial_\gamma X)\cdot(\partial_\delta X). \qquad (3.8)$$

The solution to this equation relates the dynamical worldsheet metric to the induced metric from the Nambu–Goto action via

$$g_{\alpha\beta} = f(\xi)\,(\partial_\alpha X)\cdot(\partial_\beta X) = f(\xi)\gamma_{\alpha\beta}, \qquad (3.9)$$

where $f(\xi)$ is a local scale. This scale nicely cancels in the action as well as in all equations of motion. The new redundancy introduced by describing the string with the help of the dynamical worldsheet metric is called *Weyl invariance*:

$$g_{\alpha\beta}(\xi) \mapsto f(\xi) g_{\alpha\beta}(\xi). \qquad (3.10)$$

3.3 Conformal Gauge

In the polynomial formulation, one can now make use of the gauge freedom in order simplify the e.o.m. for X. The non-linearity in the coupling to g poses a major obstruction to solve it, but we are lucky that we can remove it by demanding a conformally flat metric:

$$g_{\alpha\beta}(\xi) = f(\xi)\eta_{\alpha\beta}. \qquad (3.11)$$

This gauge choice amounts to two equations:

$$g_{\tau\sigma}(\xi) = 0 \quad \text{and} \quad g_{\tau\tau}(\xi) = -g_{\sigma\sigma}(\xi). \qquad (3.12)$$

By fixing the metric to be conformally flat, we have been using all diffeomorphisms on the worldsheet which preserve the metric up to scale f. The remaining diffeomorphisms are the Weyl scalings; in fact, one can furthermore set $f = 1$ and thus fix the Weyl redundancy as well. The resulting action describes D free massless scalar particles on the worldsheet

$$S = -\frac{1}{2\pi\kappa^2}\int d^2\xi \tfrac{1}{2}\eta^{\alpha\beta}(\partial_\alpha X)\cdot(\partial_\beta X) \qquad (3.13)$$

and the corresponding equation of motion for X is simply the harmonic wave equation:

$$\partial^2 X^\mu = 0 \quad \text{or} \quad \ddot{X} = X''. \qquad (3.14)$$

However, although having imposed conformal gauge, there is still an equation of motion for the worldsheet metric which we need to take into account (in conformal gauge):

$$T_{\alpha\beta} := (\partial_\alpha X)\cdot(\partial_\beta X) - \tfrac{1}{2}\eta_{\alpha\beta}\,\eta^{\gamma\delta}(\partial_\gamma X)\cdot(\partial_\delta X) = 0, \qquad (3.15)$$

where $T_{\alpha\beta}$ is the energy-momentum tensor for D scalar particles. The trace of the energy-momentum tensor vanishes by construction due to the conformal/Weyl symmetry ($T^\alpha_\alpha = 0$), while the two remaining e.o.m. are called *Virasoro constraints*

$$T_{\tau\sigma} = \dot{X}\cdot X' = 0, \qquad T_{\tau\tau} = T_{\sigma\sigma} = \dot{X}\cdot\dot{X} + X'\cdot X' = 0. \qquad (3.16)$$

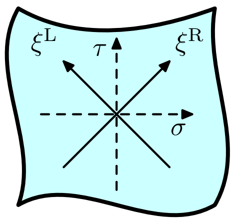
The first constraint demands the lines of constant τ to be orthogonal to lines of constant σ. This means, the string can move perpendicular to the direction it is stretched out. In other words, there are no longitudinal waves allowed, as already noted above: the string does not have an inner structure!

Because the energy-momentum tensor $T_{\alpha\beta}$ is conserved, it is sufficient to impose constraints on an initial time slice only.

3.4 Solution on the Light Cone

The harmonic wave equation can be most easily solved in light cone coordinates $\xi^{\mathrm{L/R}}$:

$$\xi^{\mathrm{L/R}} = \tau \mp \sigma, \quad \partial_{\mathrm{L/R}} = \tfrac{1}{2}(\partial_\tau \mp \partial_\sigma), \tag{3.17}$$



which imply the worldsheet metric

$$d^2s = -d\tau^2 + d\sigma^2 = -d\xi^{\mathrm{L}} d\xi^{\mathrm{R}}. \tag{3.18}$$

In terms of light cone coordinates, the e.o.m. and Virasoro constraints read

$$\partial_{\mathrm{L}}\partial_{\mathrm{R}} X^\mu = 0, \qquad (\partial_{\mathrm{L/R}} X)^2 = 0. \tag{3.19}$$

The first equation can be solved by a simple separation of variables

$$X^\mu(\xi^{\mathrm{L}}, \xi^{\mathrm{R}}) = X^\mu_{\mathrm{L}}(\xi^{\mathrm{L}}) + X^\mu_{\mathrm{R}}(\xi^{\mathrm{R}}). \tag{3.20}$$

While there are D left-movers X_{L} and D right-movers X_{R} initially, the Virasoro constraints $(\partial X_{\mathrm{R,L}})^2 = 0$ remove one left- and one right-mover. Two reparametrisations remain:

- *conformal transformations:*

$$\xi^{\mathrm{R}} \mapsto \xi'^{\mathrm{R}}(\xi^{\mathrm{R}}), \qquad \xi^{\mathrm{L}} \mapsto \xi'^{\mathrm{L}}(\xi^{\mathrm{L}}). \tag{3.21}$$

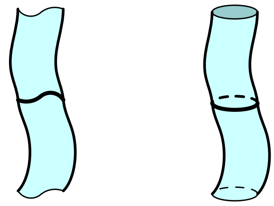
  In two dimensions (and only there), there are infinitely many conformal transformations. Those remove another left- and right-mover.
- *constant shifts:* between X^μ_{L} and X^μ_{R}.

In summary, there are $(D-2)$ left- and right-movers remaining, which parametrise the transverse directions of the string.

3.5 Closed String Modes

In the discussion so far the worldsheets have been extended infinitely in space and time. In order to have a string of finite length, let us confine the spatial extent. There are two possible basic topologies for a string:

- *closed string:* circular topology, identify $\sigma \equiv \sigma + 2\pi$ (other choices possible),
- *open string:* interval topology, boundary conditions at $\sigma = 0, \pi$ (later).



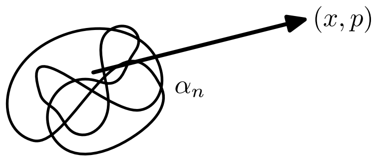
(3.22)

Covariant Formulation. For the closed string, the function function X should be 2π-periodic, so let us get started with a Fourier decomposition:

$$X^\mu_{\mathrm{L/R}} = \tfrac{1}{2} x^\mu + \tfrac{1}{2}\kappa^2 p^\mu \xi^{\mathrm{L/R}} + \sum_{n\neq 0} \frac{i\kappa}{\sqrt{2}n}\, \alpha_n^{\mathrm{L/R},\mu} \exp(-in\xi^{\mathrm{L/R}}). \tag{3.23}$$

In the above ansatz, the coefficients $i\kappa/\sqrt{2}n$ are chosen for later convenience. The linear dependency on $\xi^{\mathrm{L/R}}$ does not clash with the periodicity condition because after adding the left- and right-mover, the dependency on σ drops out: $X^\mu = x^\mu + \kappa^2 p^\mu \tau + \ldots$. Reality of the embedding X is ensured by demanding $\alpha_{-n} = (\alpha_n)^*$.

The solution exhibits two kinds of parameters. While the motion of the centre of mass is described by the conjugate pair x, p (conjugation involves an additional factor of κ^2), the string modes $\alpha_n^{\mathrm{L/R},\mu}$ (left/right movers) describe the amplitudes of the oscillations on the string.



(3.24)

Plugging the above ansatz into the Virasoro constraints $(\partial_{\mathrm{L/R}} X^\mu_{\mathrm{L/R}})^2 = 0$ yields

$$\kappa^2 \sum\nolimits_n L_n^{\mathrm{L/R}} \exp(-in\xi^{\mathrm{L/R}}) = 0, \tag{3.25}$$

where we have defined the Virasoro modes (dropping the L/R index)

$$L_n := \tfrac{1}{2} \sum\nolimits_m \alpha_{n-m} \cdot \alpha_m \tag{3.26}$$

and $\alpha_0^{\mathrm{L}} = \alpha_0^{\mathrm{R}} = \kappa p/\sqrt{2}$. Demanding validity of the Virasoro constraints is equivalent to requiring the $L_n = 0$ for all n. In particular, the Virasoro constraint $L_0 = 0$ fixes the mass of the string:

$$p^2 = -M^2, \qquad M^2 = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m \,. \tag{3.27}$$

Since all Virasoro constraints $L_n = 0$ are conserved by the e.o.m.

$$\dot{L}_n = inL_n, \tag{3.28}$$

it is sufficient to impose them on initial data only. The mass of a string depends on the mode amplitudes α. If there are no modes excited, the string behaves like a massless point particle. If there are only few and small excitations, one will encounter a light (or tiny) particle while large excitations can add up to yield a big and highly massive object.

(3.29)

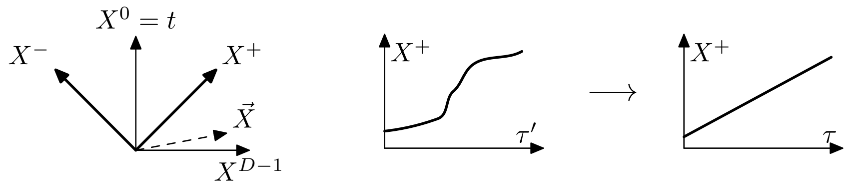
Due to the Minkowski signature of the spacetime, the time-like modes α^0 contribute a negative M^2. If one, however, takes all Virasoro constraints into account, tachyons (particles with negative mass-squared) are excluded.

Light Cone Gauge. The Virasoro constraints in their above form are complicated and non-linear. A convenient solution to this problem is to employ the conformal symmetry on the worldsheet in order to solve them. The connection between the conformal symmetry on the worldsheet and the ability to choose the particular gauge in light cone coordinates below is not spelled out here explicitly.

The main idea is to first switch to light cone coordinates in spacetime

$$X^{\pm} = X^0 \pm X^{D-1}\,, \tag{3.30}$$

where $\vec{X}$ denotes the transverse components $1 \ldots (D-2)$. Now one selects a gauge, in which the coordinate X^+ depends on τ linearly, i.e. does not exhibit oscillator modes.



(3.31)

Correspondingly, the gauge condition reads

$$X^+_{\mathrm{L/R}} = x^+_{\mathrm{L/R}} + \tfrac{1}{2}\kappa^2 p^+_{\mathrm{L/R}}\xi^{\mathrm{L/R}}. \tag{3.32}$$

In light cone coordinates, the Virasoro constraint $(\partial\vec{X}_{\mathrm{L/R}})^2 - \kappa^2 p^+_{\mathrm{L/R}}\partial X^-_{\mathrm{L/R}} = 0$ is solved by

$$X^-_{\mathrm{L/R}}(\xi) = \frac{1}{\kappa^2 p^+_{\mathrm{L/R}}}\int^{\xi} d\xi'\big(\partial\vec{X}_{\mathrm{L/R}}(\xi')\big)^2, \tag{3.33}$$

leaving us with $2(D-2)$ arbitrary functions $\vec{X}_{\mathrm{L/R}}(\xi^{\mathrm{L/R}})$. These are the transverse modes of the string.

Periodicity. Of course all functions X still have to be periodic. While the constraint from periodicity of $\vec{X}(\xi)$ is exploited in the next paragraph, periodicity of X^+ and X^- requires

$$p_{\mathrm{L}}^+ = p_{\mathrm{R}}^+ = p^+, \qquad \int_0^{2\pi} d\xi \big((\partial\vec{X}_{\mathrm{R}})^2 - (\partial\vec{X}_{\mathrm{L}})^2\big) = 0. \tag{3.34}$$

In addition, there is the residual gauge freedom $\Delta X_{\mathrm{R}}^\mu(\xi^{\mathrm{R}}) = -\Delta X_{\mathrm{L}}^\mu(\xi^{\mathrm{L}}) = \text{const.}$, which corresponds to the residual constraints mentioned above.

String Modes. Imposing gauge fixing on the modes α_n ($n \neq 0$) leads to

$$\alpha_n^+ = 0, \qquad \alpha_n^- = \frac{1}{\alpha_0^+} \sum\nolimits_m \vec{\alpha}_{n-m} \cdot \vec{\alpha}_m, \tag{3.35}$$

that is, the modes α^- are again determined in terms of α^+ and the modes $\vec{\alpha}$. Periodicity requires $\vec{\alpha}_0^{\mathrm{L}} = \vec{\alpha}_0^{\mathrm{R}}$, $\alpha_0^{\mathrm{R},+} = \alpha_0^{\mathrm{L},+}$, $\alpha_0^{\mathrm{R},+} = \alpha_0^{\mathrm{L},+}$ as well as

$$\sum_{m=1}^{\infty} \vec{\alpha}_{-m}^{\mathrm{L}} \cdot \vec{\alpha}_m^{\mathrm{L}} = \sum_{m=1}^{\infty} \vec{\alpha}_{-m}^{\mathrm{R}} \cdot \vec{\alpha}_m^{\mathrm{R}}. \tag{3.36}$$

Taking the above conditions into account, the resulting mass is manifestly positive:

$$M^2 = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} \vec{\alpha}_{-m} \cdot \vec{\alpha}_m = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} |\vec{\alpha}_m|^2, \tag{3.37}$$

where the reality condition $\alpha_{-n} = (\alpha_n)^*$ has been used in the last equation.

In summary, the light cone gauge comes with manifestly positive mass for all particles and is a very convenient way to get rid of almost all constraints. However, by introducing light cone coordinates, we give up a manifestly Lorentz-invariant formulation.

3.6 Hamiltonian Formalism

Before quantising the string in the next chapter, let us have a brief look at the classical string in the Hamiltonian formalism. We would like to derive the Poisson brackets for the variables x, p, α_n.

Fourier Modes. Let us start with the Polyakov action in conformal gauge

$$S = \int d\tau L, \qquad L = \frac{1}{2\pi\kappa^2} \int_0^{2\pi} d\sigma \tfrac{1}{2}\big(\dot{X}^2 - X'^2\big) \tag{3.38}$$

and substitute the closed-string mode expansion (with free time dependence)

$$X^\mu = \kappa \sum\nolimits_n \beta_n^\mu(\tau) \exp(-in\sigma). \tag{3.39}$$

Now calculate the derivatives and perform the integration over σ in order to obtain a tower of independent harmonic oscillators (here β_0 is the free particle):

$$L = \frac{1}{2}\sum\nolimits_n \left(\dot{\beta}_n \cdot \dot{\beta}_{-n} - n^2 \beta_n \cdot \beta_{-n}\right). \tag{3.40}$$

The Poisson bracket between canonical momentum and β reads:

$$\pi_n = \dot{\beta}_n, \qquad \{\beta_m^\mu, \pi_n^\nu\} = \eta^{\mu\nu}\delta_{m+n}. \tag{3.41}$$

Matching X with the previous classical solution at $\tau = 0$

$$x^\mu = \kappa\beta_0^\mu, \qquad p^\mu = \frac{\pi_0^\mu}{\kappa}, \qquad \alpha_n^{\mathrm{L/R},\mu} = \frac{n\beta_{\mp n}^\mu}{i\sqrt{2}} + \frac{\pi_{\mp n}^\mu}{\sqrt{2}} \tag{3.42}$$

leads to the following non-trivial Poisson brackets in the original variables

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu}, \quad \{\alpha_m^{\mathrm{L},\mu}, \alpha_n^{\mathrm{L},\nu}\} = \{\alpha_m^{\mathrm{R},\mu}, \alpha_n^{\mathrm{R},\nu}\} = -im\,\eta^{\mu\nu}\delta_{m+n}. \tag{3.43}$$

Field Theory. The same result could have been obtained borrowing some mathematical tools which are prominently used in quantum field theory. The conjugate momentum Π^μ and canonical Poisson brackets read

$$\Pi^\mu = \frac{1}{2\pi\kappa^2}\dot{X}^\mu, \qquad \{X^\mu(\sigma), \Pi^\nu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma - \sigma'). \tag{3.44}$$

Now, given those basic relations, one can use and derive the following relations:

$$\frac{1}{2\pi}\int_0^{2\pi} d\sigma X^\mu(0,\sigma) = x^\mu, \qquad \int_0^{2\pi} d\sigma \Pi^\mu(0,\sigma) = p^\mu \tag{3.45}$$

as well as

$$\begin{aligned}\int_0^{2\pi} d\sigma X^\mu(0,\sigma)\exp(-in\sigma) &= \frac{2\pi i\kappa}{\sqrt{2}n}\,(\alpha_n^{\mathrm{L},\mu} - \alpha_{-n}^{\mathrm{R},\mu}),\\ \int_0^{2\pi} d\sigma \Pi^\mu(0,\sigma)\exp(-in\sigma) &= \frac{1}{\sqrt{2}\kappa}\,(\alpha_n^{\mathrm{L},\mu} + \alpha_{-n}^{\mathrm{R},\mu}),\end{aligned} \tag{3.46}$$

where we have used the fundamental Fourier integral $(2\pi)^{-1}\int d\sigma \exp(i\sigma(n-n')) = 2\pi\delta_{n-n'}$. Using linearity of the Poisson bracket, it is not too difficult to obtain the above Poisson brackets of modes.

4 String Quantisation

We have seen that the classical closed string is described by

- a bunch of harmonic oscillators $\alpha_n^{\mathrm{L/R}}$ for the string modes;
- a relativistic particle (x, p) describing the centre of mass.

Both systems are coupled via Virasoro constraints.

We have derived two reasonable formulations:

- Covariant formulation
  - D oscillators α_n^μ per mode,
  - physical solutions must obey Virasoro constraints $L_n^{\mathrm{R}} = L_n^{\mathrm{L}} = 0$,
  - Poincaré symmetry in spacetime manifest,
  - worldsheet theory has conformal symmetry (later).
- Light cone formulation
  - $D - 2$ oscillators $\vec{\alpha}_n$ per mode,
  - physical solutions must obey $L_0^{\mathrm{R}} = L_0^{\mathrm{L}} = 0$ residual constraints,
  - Poincaré symmetry of spacetime partially manifest,
  - manifest Poincaré symmetry on worldsheet.

4.1 Canonical Quantisation

Commutation Relations. In the Hamiltonian formulation of string theory in conformal gauge we have derived the following set of non-trivial Poisson brackets between the variables x, p, α_n

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu},$$
$$\left\{\alpha_m^{\mathrm{L},\mu}, \alpha_n^{\mathrm{L},\nu}\right\} = \left\{\alpha_m^{\mathrm{R},\mu}, \alpha_n^{\mathrm{R},\nu}\right\} = -im\,\eta^{\mu\nu}\delta_{m+n}. \tag{4.1}$$

In canonical quantisation this results in the following set of commutation relations of the corresponding quantum operators[1]

$$[x^\mu, p^\nu] = i\eta^{\mu\nu},$$
$$\left[\alpha_m^{\mathrm{L},\mu}, \alpha_n^{\mathrm{L},\nu}\right] = \left[\alpha_m^{\mathrm{R},\mu}, \alpha_n^{\mathrm{R},\nu}\right] = m\,\eta^{\mu\nu}\delta_{m+n}. \tag{4.2}$$

Space of States. Compose the space of states from a quantum free particle and a set of quantum harmonic oscillators:

- momentum eigenstates for free particle $|q\rangle$,
- HO vacuum $|0\rangle$ and excitations for each mode/orientation.

[1]We shall use the same symbols for classical variables and corresponding quantum operators. The precise meaning should be clear from the context.

Define string vacuum state $|0;q\rangle$

$$p^\mu|0;q\rangle = q^\mu|0;q\rangle, \qquad \alpha_n^{\mathrm{L/R},\mu}|0;q\rangle = 0 \quad \text{for } n>0. \tag{4.3}$$

Negative Norm States. One problem: we have states with negative norm

$$\begin{aligned}&|n,\mu,\mathrm{L/R};q\rangle := \alpha_{-n}^{\mathrm{L/R},\mu}|0;q\rangle,\\ &|n,\mu,\mathrm{L/R};q|^2 = \langle 0;q|\alpha_n^{\mathrm{L/R},\mu}\alpha_{-n}^{\mathrm{L/R},\mu}|0;q\rangle = n\eta^{\mu\mu}.\end{aligned} \tag{4.4}$$

For $\mu=0$ this state has negative norm. In fact, it is not allowed by Virasoro constraints.

General resolution of this problem: impose Virasoro constraints! All states obeying the Virasoro constraints have non-negative norm. This problem needs some care. We shall continue the covariant quantisation later.

4.2 Light Cone Quantisation

For simplicity we fix the light cone gauge; this leaves only physical states. Resulting commutators evidently lead to positive-definite states

$$\big[\alpha_m^{\mathrm{L},a},\alpha_n^{\mathrm{L},b}\big] = \big[\alpha_m^{\mathrm{R},a},\alpha_n^{\mathrm{R},b}\big] = m\,\delta^{ab}\delta_{m+n}. \tag{4.5}$$

Recall classical mass and residual constraint $L_0^{\mathrm{R}} = L_0^{\mathrm{L}} = 0$

$$M^2 = \frac{4}{\kappa^2}\sum_{m=1}^{\infty}\vec\alpha_{-m}^{\mathrm{L}}\cdot\vec\alpha_m^{\mathrm{L}} = \frac{4}{\kappa^2}\sum_{m=1}^{\infty}\vec\alpha_{-m}^{\mathrm{R}}\cdot\vec\alpha_m^{\mathrm{R}}. \tag{4.6}$$

In the quantum theory, the ordering of operators matters! A priori we are free to choose the operator ordering.[2] We therefore assume normal ordering (negative mode numbers to the left of positive mode numbers) plus two new constants $a^{\mathrm{L/R}}$:

$$M^2 = \frac{4}{\kappa^2}\left(\sum_{m=1}^{\infty}\vec\alpha_{-m}^{\mathrm{L}}\cdot\vec\alpha_m^{\mathrm{L}} - a^{\mathrm{L}}\right) = \frac{4}{\kappa^2}\left(\sum_{m=1}^{\infty}\vec\alpha_{-m}^{\mathrm{R}}\cdot\vec\alpha_m^{\mathrm{R}} - a^{\mathrm{R}}\right). \tag{4.7}$$

The term involving the oscillators α measures the so-called string level (which is a non-negative integer in the quantum theory)

$$N := \sum_{m=1}^{\infty}\vec\alpha_{-m}\cdot\vec\alpha_m = \sum_{m=1}^{\infty} mN_m \quad \text{with } N_m := \frac{1}{m}\,\vec\alpha_{-m}\cdot\vec\alpha_m. \tag{4.8}$$

The mass and residual constraint can be expressed in terms of string level

$$M^2 = \frac{4}{\kappa^2}(N^{\mathrm{L}} - a^{\mathrm{L}}) = \frac{4}{\kappa^2}(N^{\mathrm{R}} - a^{\mathrm{R}}). \tag{4.9}$$

[2]Quantisation of a classical model is not necessarily a unique procedure. Typically it requires the introduction of additional parameters which parametrise the arbitrariness (or our ignorance). These parameters are of order $\hbar$ when taking the classical limit, and therefore have no classical counterpart.

4.3 String Spectrum

The mass of the string state depends on string level. Quantisation of string level leads to a quantisation of mass for string states. Level matching:

$$N^{\mathrm{L}} - a^{\mathrm{L}} = N^{\mathrm{R}} - a^{\mathrm{R}}. \tag{4.10}$$

Let us understand string states at each level: HO's.

Vacuum State. Define vacuum state $|0; q\rangle$

$$\vec{\alpha}_n^{\mathrm{L/R}} |0; q\rangle = 0 \quad \text{for } n > 0. \tag{4.11}$$

This state is at level zero: $N^{\mathrm{L}} = N^{\mathrm{R}} = 0$. It has spin zero.[3]

For a physical state we must require:

$$a^{\mathrm{R}} = a^{\mathrm{L}} = a, \qquad M^2 = -\frac{4a}{\kappa^2}\,. \tag{4.12}$$

So far so good: We have a spin-0 particle with $M = 2\kappa^{-1}\sqrt{-a}$. For a physical particle we would have to require $a \leq 0$! Spatial extent of state: HO wave function $\sim \kappa$.

First Level. Lowest excited state has $N = 1$. Level matching and $a^{\mathrm{L}} = a^{\mathrm{R}}$ implies $N^{\mathrm{L}} = N^{\mathrm{R}} = 1$. One excitation $\vec{\alpha}_{-1}$ each from left/right movers

$$|ab; q\rangle = \alpha_{-1}^{\mathrm{L},a} \alpha_{-1}^{\mathrm{R},b} |0; q\rangle. \tag{4.13}$$

We find $(D-2)^2$ states of mass $M = 2\kappa^{-1}\sqrt{1-a}$.

What is their spin? Consider first the spin under $\mathrm{SO}(D-2)$ transverse rotations. There are three useful combinations:

$$\begin{aligned} |(ab); q\rangle &:= |ab; q\rangle + |ba; q\rangle - \frac{2\delta_{ab}}{D-2} |cc; q\rangle, \\ |[ab]; q\rangle &:= |ab; q\rangle - |ba; q\rangle, \\ |1; q\rangle &:= |cc; q\rangle. \end{aligned} \tag{4.14}$$

Transformation properties under $\mathrm{SO}(D-2)$:

state	indices	Young tab.	"spin"
$\vert(ab); q\rangle$	symmetric, traceless	□□	2
$\vert[ab]; q\rangle$	anti-symmetric	⊟	1
$\vert1; q\rangle$	singlet	•	0

(4.15)

[3]This can be derived by applying the Noether charge of spacetime Poincaré symmetry.

Compare this to classification of unitary representations of the Poincaré group.[4] Stabiliser (little group) for massive particle is $\mathrm{SO}(D-1)$. Can we fit these $\mathrm{SO}(D-2)$ representation into $\mathrm{SO}(D-1)$ representations? No!

The only way out: consider a massless particle where the stabiliser is $\mathrm{SO}(D-2)$. We have to set $a = a^{\mathrm{R}} = a^{\mathrm{L}} = 1$.

Three types of particles:

- $|(\boldsymbol{ab});\boldsymbol{q}\rangle$**:** massless spin-2 field. this is fine as free field.
  Weinberg–Witten theorem: interactions are forbidden except for gravitational interactions. This particle must be the graviton!
- $|[\boldsymbol{ab}];\boldsymbol{q}\rangle$**:** massless 2-form field (Kalb–Ramond).
  generalisation of electromagnetic field A_μ: $B_{\mu\nu}$ with 1-form gauge symmetry $\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$.
- $|\mathbf{1};\boldsymbol{q}\rangle$**:** massless scalar particle (dilaton).
  state different from string vacuum $|0;q\rangle$.

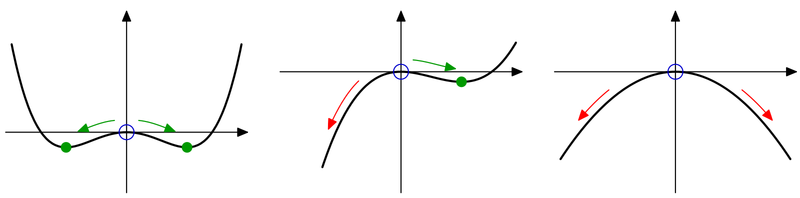
What we have learned so far:

- String theory contains graviton plus massless 2-form and scalar particles.
- Interacting string theory includes gravity!
- Spatial extent of particles $\sim \kappa$; practically point-like.
- κ is the Planck scale (later).
- We must set $a = a^{\mathrm{R}} = a^{\mathrm{L}} = 1$ for consistency (with physicality).

Tachyon. Now revisit the string vacuum $|q,0\rangle$: $M^2 = -4/\kappa^2 < 0$. This state is a tachyon!

Is this a problem? Not really, compare to spontaneous symmetry breaking mechanism:

- Vacuum state was chosen at unstable local maximum of a potential. This leads to a tachyonic mode.
- Physical ground state should be situated at local minimum. No tachyon here!
- (Bosonic) string theory: Unclear if a global minimum exists. Unclear how to treat it in practice: Where is it? What are its properties?



(4.16)

- Let us ignore this shortcoming. Indeed, the tachyon is absent for superstrings (which are treated later)!

[4]This is a topic of QFT I: In short, massive particles are characterised by their spin (representation of $\mathrm{SO}(D-1)$ stabiliser group of massive particle trajectory) while massless particles are characterised by their helicity (representation of $\mathrm{SO}(D-2)$ stabiliser group of massless particle trajectory). Use the well-known case $D=4$ for comparison.

Higher Levels. Levels zero and one work out. What about the higher levels? We have already used up all available freedom to adjust $a^{\mathrm{L,R}}$, now self-consistency of the model is up to luck (or faith). The following table lists the representations for left-movers (equivalently right-movers)

level	excitations	$\mathrm{SO}(D-2)$	$\mathrm{SO}(D-1)$
0	$\cdot$	$\bullet$	$\bullet$
1	α^a_{-1}	$\yng(1)$	$\times$
2	$\alpha^a_{-1}\alpha^b_{-1}$	$\yng(2)+\bullet$	$\yng(2)$
	α^a_{-2}	$\yng(1)$	
3	$\alpha^a_{-1}\alpha^b_{-1}\alpha^c_{-1}$	$\yng(3)+\yng(1)$	$\yng(3)$
	$\alpha^a_{-1}\alpha^b_{-2}$	$\yng(2)+\yng(1,1)+\bullet$	$\yng(1,1)$
	α^a_{-3}	$\yng(1)$	
4	$\alpha^a_{-1}\alpha^b_{-1}\alpha^c_{-1}\alpha^d_{-1}$	$\yng(4)+\yng(2)+\bullet$	$\yng(4)$
	$\alpha^a_{-1}\alpha^b_{-1}\alpha^c_{-2}$	$\yng(3)+\yng(2,1)+\yng(1)+\yng(1)$	$\yng(2,1)$
	$\alpha^a_{-2}\alpha^b_{-2}$	$\yng(2)+\bullet$	$\yng(2)$
	$\alpha^a_{-3}\alpha^b_{-1}$	$\yng(2)+\yng(1,1)+\bullet$	$\bullet$
	α^a_{-4}	$\yng(1)$	
...	...	...	...

(4.17)

All higher levels combine into proper $\mathrm{SO}(D-1)$ representations, for both left and right moving modes.

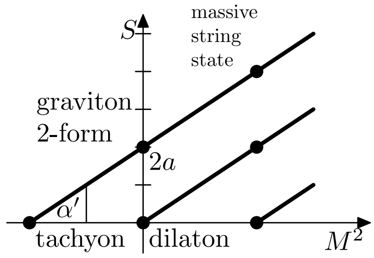
Furthermore, level matching implies we need to square the above representations for the correct particle spectrum.

Altogether:

- String describes collection of infinitely many particle types of different mass and spin.
- Various vibration modes might correspond to elementary particles. They include the massless graviton.
- Intrinsic particle extent at Planck scale κ. Planck scale is much smalled than can be observed: String theory describes practically point-like particles!
- String theory describes only a few massless particles; all others at Planck mass $1/\kappa$; one tachyon.
- Proper treatment of tachyon could change picture altogether.
- Very high excitations are long strings. They would mostly display classical behaviour, but are superheavy $M \gg 1/\kappa$.

Regge Trajectories. Maximum spin (all indices symmetric and traceless) increases linearly with level $S = 2N$

$$M^2 = \frac{2S-4a}{\kappa^2},$$



(4.18)

This relationship is called the leading “Regge trajectory”:

- $\alpha' = \kappa^2$ is called the Regge slope.
- $2a$ is called the Regge intercept; spin of massless particle.

Subleading trajectories have lower spins (indices anti-symmetric or have trace).

Qualitative similarity to hadron spectrum:

- Regge trajectories observed for hadronic resonances.
- For a stringy description of QCD, $1/\kappa$ would have to be QCD scale $\simeq 1$ GeV.
- Intercept should be $a \approx -\frac{1}{2}$ for QCD rather than $a = 1$.
- There is another problem (see later).
- Strings provide a qualitative description of QCD flux tubes.

4.4 Anomalies

In light cone gauge we have broken manifest $\mathrm{SO}(D-1,1)$ Lorentz symmetry to a $\mathrm{SO}(D-2)$ subgroup.

- Consequently the spectrum of quantum strings organises manifestly into $\mathrm{SO}(D-2)$ multiplets.
- Almost all multiplets fit into $\mathrm{SO}(D-1)$ multiplets.
- Mass assignments fill Poincaré multiplets for $a^{\mathrm{L}} = a^{\mathrm{R}} = 1$.
- Poincaré symmetry broken unless $a^{\mathrm{L}} = a^{\mathrm{R}} = 1$.

Anomaly: Failure of classical symmetry in quantum theory.

Sometimes anomalies are permissible, but not here because we want strings to propagate on a Minkowski background with intact Poincaré symmetry.

So far we have only done counting, a more severe problem exists in the algebra. The commutator $[M^{-a}, M^{-b}]$ is supposed to vanish, but it receives contributions from $[\alpha^-, \alpha^a]$ which is non-zero in light cone gauge due to the solution of α^- in terms of an integral. One finds

$$[M^{-a}, M^{-b}] = \sum_{n=1}^{\infty} \left[\left(\frac{D-2}{24} - 1 \right) \left(n - \frac{1}{n} \right) + \frac{a-1}{n} \right] \times \ldots \tag{4.19}$$

This expression vanishes if and only if $D = 26$ and $a = 1$. String theory predicts twenty-six spacetime dimensions.

There is a shortcut derivation: reconsider the nature of the intercept a as a sum of HO ground state energies $\frac{1}{2}\omega_n = \frac{1}{2}n$

$$a = -\sum_{n=1}^{\infty} (D-2)\tfrac{1}{2}\omega_n = -\tfrac{1}{2}(D-2)\sum_{n=1}^{\infty} n. \tag{4.20}$$

This sum is divergent, but black magic helps: ζ-function regularisation

$$\zeta(z) := \sum_{k=1}^{\infty} \frac{1}{k^z}, \quad \text{i.e.} \quad a = -\tfrac{1}{2}(D-2)\zeta(-1) = \frac{D-2}{24}\,. \tag{4.21}$$

Analytical continuation $\zeta(-1) = -\frac{1}{12}$ and use of $a = 1$ predicts $D = 26$! Here a somewhat questionable derivation yields the correct prediction.

4.5 Covariant Quantisation

In LC gauge Poincaré symmetry is subject to an anomaly, but we can also keep Poincaré symmetry manifest: Covariant quantisation. Let us see how the spectrum arises in covariant approach. For simplicity, consider only one set of L or R oscillators.

Vacuum State. The vacuum state $|0;q\rangle$ is defined as before by $\alpha_0^\mu|0;q\rangle \sim \kappa q^\mu|0;q\rangle$, $\alpha_{n>0}^\mu|0;q\rangle = 0$. It satisfies

$$L_{n>0}|0;q\rangle = 0 \quad \text{and} \quad L_0|0;q\rangle = \frac{\kappa^2 q^2}{4}\,|0;q\rangle. \tag{4.22}$$

The state is not annihilated by the negative Virasoro modes. Instead $\langle 0;q|L_{n<0} = 0$ hence $\langle 0;q|(L_n - \delta_n a)|0;q\rangle = 0$.

For generic physical states $|\mathit{\Psi}\rangle$, $\langle\mathit{\Phi}|$ we should impose the Virasoro constraints[5]

$$\begin{aligned} L_{n>0}|\mathit{\Psi}\rangle &= 0, & L_0|\mathit{\Psi}\rangle &= a|\mathit{\Psi}\rangle, \\ \langle\mathit{\Phi}|L_{n<0} &= 0, & \langle\mathit{\Phi}|L_0 &= a|\mathit{\Phi}\rangle, \end{aligned} \tag{4.23}$$

such that we have

$$\langle\mathit{\Phi}|(L_n - \delta_n a)|\mathit{\Psi}\rangle = 0. \tag{4.24}$$

One Excitation. We make a generic ansatz for the one-excitation state

$$|\psi;q\rangle := \psi\cdot\alpha_{-1}|0;q\rangle. \tag{4.25}$$

The norm $\bar{\psi}\cdot\psi$ is potentially negative. The level-one Virasoro constraint acting on the state implies

$$L_1|\psi;q\rangle = \alpha_1\cdot\alpha_0|\psi;q\rangle = \frac{\kappa(\psi\cdot q)}{\sqrt{2}}\,|0;q\rangle = 0. \tag{4.26}$$

Furthermore $L_0 = a = 1$ implies $q^2 = 0$. In this case, the constraint $q\cdot\psi = 0$ removes the negative norm state. The following remains:

- $D-2$ states with positive norm.
- One state with $\psi \sim q$. This state has vanishing norm, it is “null”. Null states do (should) not contribute to physics.

[5]This is analogous to the Gupta–Bleuler quantisation of the electromagnetic field where the gauge fixing condition is imposed on scalar products of states.

Two Excitations. Now start with the generic ansatz

$$|\phi, \psi; q\rangle := \phi_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}|0; q\rangle + \psi_{\mu}\alpha^{\mu}_{-2}|0; q\rangle. \tag{4.27}$$

Then impose the constraints $L_0, L_1, L_2 = 0$ to fix $q^2, \psi, \operatorname{tr}\phi$. What remains is: $\square\!\square$, $\square$, $\bullet$ of $\mathrm{SO}(D-1)$.[6]

- The ansatz for $\square\!\square$ uses a generic symmetric traceless tensor $\phi_{\mu\nu}$ with $\phi_{\mu\nu}q^{\nu} = 0$. This state is positive definite.
- Ansatz for $\square$: $\phi_{\mu\nu} = q_{\mu}\rho_{\nu} + \rho_{\mu}q_{\nu}$ with $q \cdot \rho = 0$. Negative norm for $1 < a < 2$; null for $a = 1$!
- Ansatz for $\bullet$: $\phi_{\mu\nu} = q_{\mu}q_{\nu} + \eta_{\mu\nu}\sigma$. Negative norm for $D < 1$ or $D > 26$; null for $D = 26$!

Virasoro Algebra. The algebra of quantum charges L_n reads

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}. \tag{4.28}$$

This defines the Virasoro algebra. The latter term is the central charge c of Virasoro algebras (rather its representation on fields), namely $c = D$ in our case. We are interested in primary states

$$L_{n>0}|\Psi\rangle = 0, \quad L_0|\Psi\rangle = a|\Psi\rangle. \tag{4.29}$$

We can now apply representation theory of the Virasoro algebra $\Rightarrow$ 2D CFT.

A proper treatment (BRST) requires additional ghost fields. Inclusion of ghost fields recovers the classical conformal algebra precisely when $D = 26$ and $a = 1$

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{4.30}$$

In covariant gauge, the conformal algebra is anomalous. In light cone gauge, however, the anomaly is shifted to the Lorentz algebra.[7] The anomaly is proportional to $D - 26$ and $a - 1$, i.e. it can be avoided for a very specific choice of parameters.

[6]Orthogonality w.r.t. the timelike vector q effectively reduces vector indices from $\mathrm{SO}(D-1,1)$ to $\mathrm{SO}(D-1)$.

[7]The anomaly actually affects the product of the Virasoro and the Poincaré algebra, but neither of the individual algebras alone.

5 Compactification and T-Duality

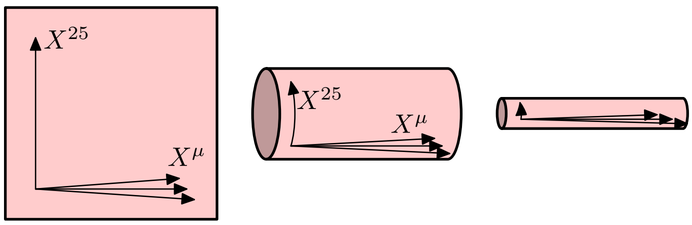
We have seen that the closed string spectrum contains:

- one tachyonic scalar particle (due to wrong vacuum),
- a graviton and few other massless particles,
- a tower of particles of increasing mass (practically inaccessible due to Planck scale masses).

But: string theory requires $D = 26$ dimensions for consistency; that is way too many to match with reality! In particular, the Gauss law predicts that the gravitational force F decays with a high power of the distance r namely $F \sim 1/A \sim 1/r^{24}$ instead of the observed $F \sim 1/r^2$ behaviour.

5.1 Kaluza–Klein Modes

Idea: Compactify 22 spatial dimensions to a microscopic size. Large distances exist only along the 4 remaining spacetime dimensions. For a sufficiently small compactification radius, the compact dimensions are almost unobservable.



(5.1)

Compactify the coordinate X^{25} to a circle of radius R

$$X^{25} \equiv X^{25} + 2\pi R. \tag{5.2}$$

The quantum mechanical momentum now becomes quantised

$$P_{25} = \frac{n}{R}\,. \tag{5.3}$$

Effectively, we obtain a tower of massive particles $M_{25}^2 = M_{26}^2 + n^2/R^2$:[1]

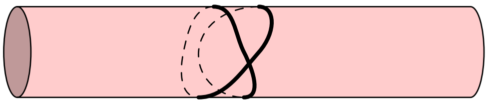
- The zero mode $n = 0$ has the original mass. This mode can be observable if the original mass is sufficiently small or massless.
- Higher modes are massive, $M \simeq 1/R$. For sufficiently small radius R these modes are practically unobservable.

For 22 compact dimensions with a sufficiently small compactification radius, the low-energy physics can be effectively four-dimensional.

[1] M_{25} denotes the effective mass of a particle propagating in the 25 non-compact dimensions whereas M_{26} denotes the original mass of the particle propagating in all 26 dimensions.

5.2 Winding Modes

Strings on compact spaces have a new peculiarity: winding modes.



(5.4)

Consider again one compact direction $X^{25} =: X \equiv X + 2\pi R$. We need to adapt the periodicity condition for the closed string to the needs of the compact dimension

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi R m. \tag{5.5}$$

Here, m is called the *winding number*. This periodicity condition implies the following expansion in terms of string modes

$$X_{\mathrm{L/R}} = \tfrac{1}{2}x + \tfrac{1}{2}\kappa^2 \left(\frac{n}{R} \mp \frac{mR}{\kappa^2}\right) \xi^{\mathrm{L/R}} + \text{modes}. \tag{5.6}$$

Mass Spectrum. The resulting effective mass (for propagation in 25 non-compact dimensions) reads

$$M^2 = \frac{4}{\kappa^2}(N^{\mathrm{L/R}} - a) + \left(\frac{n}{R} \mp \frac{mR}{\kappa^2}\right)^2 . \tag{5.7}$$

The level matching condition is modified by winding[2]

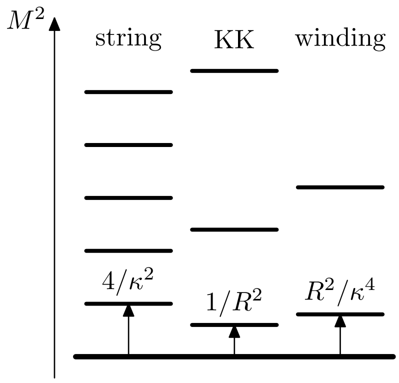
$$N^{\mathrm{L}} - N^{\mathrm{R}} = nm. \tag{5.8}$$

We can use it to rewrite the mass formula as an average over the L/R level numbers

$$M^2 = \frac{2}{\kappa^2}(N^{\mathrm{L}} + N^{\mathrm{R}} - 2a) + \frac{n^2}{R^2} + \frac{m^2 R^2}{\kappa^4} . \tag{5.9}$$

This clearly shows that winding contributes (positively) to the mass much like the KK modes.

In order to hide all but finitely many string modes: all of κ, R and κ^2/R must be small.



(5.10)

Therefore a complete compactification $R \to 0$ does not lead to the desired effect, the compactification radius should be of the order of κ.

[2] Also modes with $N^{\mathrm{L}} \neq N^{\mathrm{R}}$ can now exist. This leads to new types of spin representations for the closed string.

Compactification and Decompactification. We can now investigate what happens when we "decompactify" the circle as $R \to \infty$:

- Winding modes become very heavy.
- KK modes form become light and form a continuum.

(5.11)

The additional winding modes become infinitely heavy as $R \to \infty$. This makes sense because a string with winding becomes very long and the tension is responsible for a large mass. Furthermore, the low-energy spectrum reproduces the string spectrum without compactification.

Alternatively, we can investigate the compactification limit $R \to 0$:

- KK modes become very heavy.
- Winding modes become light and form continuum.

We obtain the same picture as for $R \to \infty$ with role of m and n interchanged. Moreover, we observe that the spectrum is the precisely the same for R and κ^2/R.

We conclude that the additional dimension remains observable at $R \to 0$! This is very different from a regular point particle which has KK modes only.

5.3 T-Duality

We have seen a duality between a small and a large compactification radius. We can in fact show this feature at Lagrangian level: *T-duality*.

We start with the string action of the 25-direction $X := X^{25}$ in conformal gauge

$$-\frac{1}{2\pi\kappa^2}\int d^2\xi\, \tfrac{1}{2}\eta^{ab}\,\partial_a X\,\partial_b X. \tag{5.12}$$

The action has a global shift symmetry $X \to X + \epsilon$. The winding modes could be viewed as a shift by $\epsilon = 2\pi m R$ which is localised to the boundary. Let us therefore make the symmetry local by introducing a new gauge field A_a

$$\frac{1}{2\pi\kappa^2}\int d^2\xi \left(-\tfrac{1}{2}\eta^{ab}(\partial_a X + A_a)(\partial_b X + A_b) - \varepsilon^{ab}\tilde{X}\partial_a A_b\right). \tag{5.13}$$

This action is now invariant under a local shift symmetry $X \to X + \epsilon$ together with $A_a \to A_a - \partial_a\epsilon$. We have added two d.o.f. in the field A_a and one local redundancy in the shift by ϵ. We remove the remaining additional d.o.f. by demanding $F_{ab} = \partial_a A_b - \partial_b A_a = 0$ through a Lagrange multiplier $\tilde{X}$. This implies that locally the gauge field A_a is trivial, i.e. it is gauge equivalent to $A_a = 0$. The new action is therefore equivalent to the old one, we have neither gained nor lost anything, but it enables us to perform the following step.

The field A_a has no derivatives in the action, it is algebraic, and we can integrate it out exactly. The e.o.m. read

$$A_a = -\partial_a X - \eta_{ac}\varepsilon^{cb}\partial_b \tilde{X}. \tag{5.14}$$

We substitute them into the action and obtain a new action (up to a boundary term)

$$-\frac{1}{2\pi\kappa^2}\int d^2\xi\,\tfrac{1}{2}\eta^{ab}\partial_a\tilde{X}\,\partial_b\tilde{X}. \tag{5.15}$$

This action has the same form as before, but it is expressed in terms of $\tilde{X}$ instead of X.

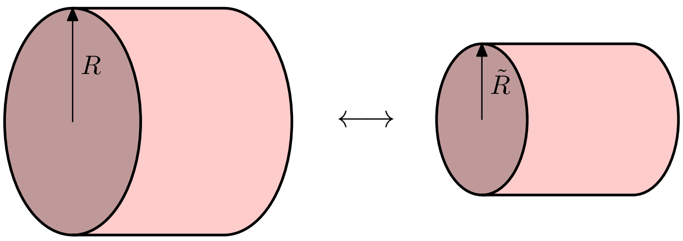
Now we can set $A_a = 0$ to go back to the original formulation, and obtain the duality relation

$$\partial_a X = -\eta_{ac}\varepsilon^{cb}\partial_b\tilde{X}, \quad \text{i.e.} \quad \dot{X} = \tilde{X}', \quad X' = \dot{\tilde{X}}. \tag{5.16}$$

This transformation is called T-duality.[3] Since the action in the dual coordinates is the same as before, solutions are mapped to solutions (potentially modulo boundary conditions). For the standard solution X we find the dual solution $\tilde{X}$

$$\begin{aligned} X &= x + \kappa^2\frac{n}{R}\tau + mR\sigma + \text{modes}, \\ \tilde{X} &= \tilde{x} + mR\tau + \kappa^2\frac{n}{R}\sigma + \text{modes}. \end{aligned} \tag{5.17}$$

The duality interchanges the radii $R \leftrightarrow \tilde{R} = \kappa^2/R$ as well as the winding and KK numbers $m \leftrightarrow n$.



(5.18)

Therefore T-duality relates string theory on two backgrounds with different compactification radii.

This implies that $R = \kappa$ is effectively the minimum compactification radius. It is indeed a special "self-dual" point. For this special point, the duality between two models turns into an enhanced symmetry of a single model.

$R = \kappa$ is the minimum length scale in string theory:[4] This may hint at a quantisation of spacetime in quantum gravity.

[3]T-duality is a non-local transformation: Although the map between $\partial_a X$ and $\partial_b\tilde{X}$ is local, the fields X and $\tilde{X}$ are related non-locally.

[4]Recall that the HO wave function of the string vacuum has a spatial extent of $\sim \kappa$.

5.4 General Compactifications

So far we have compactified one dimension: either a circle or an interval. There are many choices and parameters for a simultaneous compactification of several dimensions, e.g.:

- sphere S^n,
- product of spheres $S^a \times S^{n-a}$ with different radii,
- torus T^n, $3n - 3$ moduli (radii, tilts),
- other compact manifolds.

The low-lying modes are determined by this manifold:[5]

- Compactification determines observable spectrum of low-energy particles.
- Goal: find a suitable manifold to describe the standard model of particle physics as a low-energy limit.
- Massless modes correspond to gauge symmetries.
- Superstrings: compactification typically breaks supersymmetry. In order to preserve one supersymmetry, the compactification manifold must be a Calabi–Yau 3-fold.[6]

[5] An analogy is the shape of a bell which determines its characteristic spectrum.
[6] This is the reason why Calabi–Yau manifolds are of interest to string theorists.

Introduction to String Theory — Chapter 6

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22. 12. 2013

6 Open Strings and D-Branes

So far we have discussed closed strings. The alternative choice is open boundary conditions which leads to some interesting new objects in string theory.

6.1 Neumann Boundary Conditions

For open strings we conventionally choose the range of the spatial coordinate σ as $0 \leq \sigma \leq \pi$, and we should discuss the boundaries at $\sigma = 0, \pi$. We start again in conformal gauge

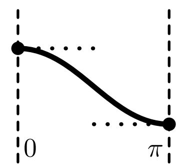
$$S = \frac{1}{2\pi\kappa^2} \int d^2\xi \, \tfrac{1}{2} \eta^{ab} \, \partial_a X \cdot \partial_b X. \tag{6.1}$$

We vary the action paying attention to boundary terms due to partial integrations

$$\begin{aligned} \delta S &= \frac{1}{2\pi\kappa^2} \int d^2\xi \, \eta^{ab} \, \partial_a \delta X \cdot \partial_b X \\ &= \frac{1}{2\pi\kappa^2} \int d^2\xi \, \partial_a (\eta^{ab} \, \delta X \cdot \partial_b X) - \ldots \\ &= \frac{1}{2\pi\kappa^2} \int d\tau \big(\delta X(\pi) \cdot X'(\pi) - \delta X(0) \cdot X'(0) \big) - \ldots . \end{aligned} \tag{6.2}$$

The boundary e.o.m. imply Neumann conditions where the function has no slope at the boundary[1]

$$X'(0) = X'(\pi) = 0. \tag{6.3}$$



The Virasoro constraints $X' \cdot \dot{X} = X'^2 + \dot{X}^2 = 0$ imply that the end points move at speed of light[2]

$$\dot{X}^2 = 0. \tag{6.4}$$

6.2 Solutions and Spectrum

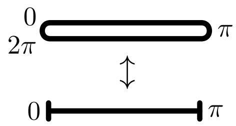
The string has the same e.o.m. in the bulk, so we can recycle the bulk solution.

[1] There is in fact an alternative implication which we shall discuss later.

[2] In the euclidean version of the string, a static soap film cannot have a free boundary: $\dot{X}^2 = 0$ implies $\dot{X} = 0$ which is in contraction with a boundary.

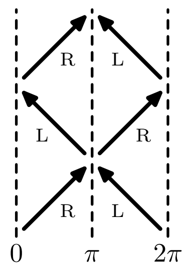
Doubling Trick. We can even relate the open string to a closed one. We map two copies of the open string to a closed string which is twice as long, and we identify the points

$$\sigma \equiv 2\pi - \sigma. \tag{6.5}$$



We then impose the boundary condition $X' = 0$ at $\sigma = 0, \pi$ on the closed string solution derived earlier. This implies[3]

$$\alpha_n^{\mathrm{L}} = \alpha_n^{\mathrm{R}} =: \alpha_n. \tag{6.6}$$



Left movers are reflected into right movers at boundaries. Therefore, we are left with only one copy of string oscillators and one copy of the Virasoro algebra. The general solution for the open string reads[4]

$$X^\mu = x^\mu + 2\kappa^2 p^\mu \tau + \sum_{n\neq 0} \frac{i\kappa}{\sqrt{2}n}\, \alpha_n^\mu \left(\exp(-in\xi^{\mathrm{L}}) + \exp(-in\xi^{\mathrm{R}})\right). \tag{6.7}$$

Quantisation. Quantisation is analogous to closed strings. We find the very same anomaly conditions $D = 26$, $a = 1$ arising from the bulk. The resulting mass spectrum reads[5]

$$M^2 = \frac{1}{\kappa^2}\,(N - a). \tag{6.8}$$

We now have only a single copy of oscillators at each level. The discussion is the same as for the closed string, but without the squaring of representations. We find:

- level 0: singlet tachyon (of half “mass”).
- level 1: massless vector: Maxwell field.
- level 2: massive spin-2 field $\square\!\square$.
- …

The massless modes at level one are associated to local symmetries:

- open string: spin-1 gauge fields,
- closed string: spin-2 gravitational fields and others.

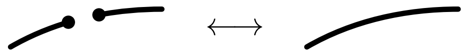
[3]In fact both boundaries yield the same set of constraints. The doubling trick is an ansatz which identifies the two boundary conditions; different boundary conditions would require a modification of the doubling trick.

[4]Note that the prefactor of the momentum p is twice as large compared to the closed string solution. This choice compensates for the halving of the integration domain in σ so that p is the Noether charge for target space translations in both cases.

[5]Note the different prefactor related to the alternative definition of p.

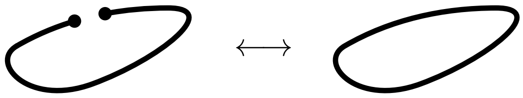
String Interactions. Let us briefly discuss interacting strings. In general, open and closed strings interact:

- Interactions of open strings certainly involves that two ends of string can join.



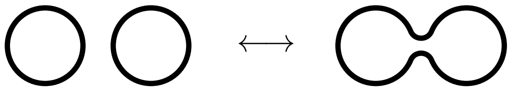
(6.9)

  When the two ends belong to a single open string, this string closes.



(6.10)

  Therefore an interacting theory of open strings must include closed strings. This is achieved by several string "vacuum" states in the Hilbert space in same theory, e.g. $|0;q\rangle_{\mathrm{c}}$ and $|0;q\rangle_{\mathrm{o}}$.
- The opening of strings can be suppressed. Closed strings can live on their own with interactions splitting or joining stings.



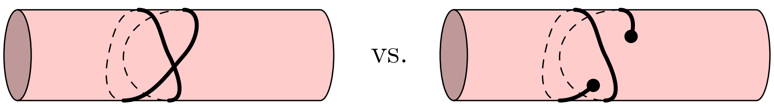
(6.11)

We conclude that string theory necessarily contains gravitational d.o.f.; It may or may not include gauge field(s) from open string sectors.

6.3 Dirichlet Boundary Conditions

The Neumann boundary conditions discussed above are not sufficient, Dirichlet boundaries should also be present in string theory as following discussion suggests.

Now consider compactification for open strings. Again, this is almost the same as for closed strings. However, we have no winding modes because the open string can always unwind.



(6.12)

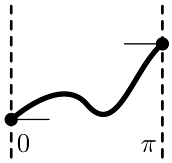
T-Duality. We apply T-duality to open strings and introduce the dual field $\tilde{X}^{25}$ by the relation

$$X'^{25} = \dot{\tilde{X}}^{25}, \qquad \dot{X}^{25} = \tilde{X}'^{25}. \tag{6.13}$$

The boundary conditions translate to

$$X'^{\mu} = 0 \quad (\mu = 0, \ldots, 24), \qquad \dot{\tilde{X}}^{25} = 0. \tag{6.14}$$

We obtain Dirichlet boundary condition for the dual coordinate $\tilde{X}^{25}$ while the Neumann boundary conditions remains for the remaining original coordinates X^{μ}. The Dirichlet condition implies that the ends of the string are fixed to constant values in the coordinate $\tilde{X}^{25}$. This corresponds to the alternate choice of boundary e.o.m. where the variation is suppressed, $\delta\tilde{X}^{25} = 0$.

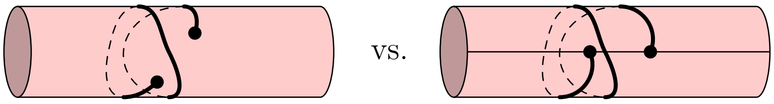


(6.15)

We can show that the dual string actually starts and ends at the same $\tilde{X}^{25}$ coordinate

$$\Delta\tilde{X}^{25} = \int d\sigma\,\tilde{X}'^{25} = \int d\sigma\,\dot{X}^{25} = 2\pi\kappa^2 p_{25} = \frac{2\pi\kappa^2 n}{R} = 2\pi n\tilde{R}. \tag{6.16}$$

Here we have used that the momentum $p_{25} = n/R$ is quantised in KK modes. The end points are then identified by the compactification of the dual coordinate $\tilde{X}^{25} \equiv \tilde{X}^{25} + 2\pi\tilde{R}$.



(6.17)

Note that the Dirichlet boundary condition constrains the string end points which allows winding modes but prevents momentum flow along $\tilde{X}^{25}$ and thus KK modes. In conclusion, T-duality exchanges the roles of KK and winding modes just as for closed strings.

The Dirichlet condition modifies the oscillator relation

$$\tilde{\alpha}_n^{\mathrm{L},25} = -\tilde{\alpha}_n^{\mathrm{R},25} =: \tilde{\alpha}_n^{25} \tag{6.18}$$

such that the general solution reads

$$\tilde{X}^{25} = \tilde{x}_0^{25} + \sum_{n\neq 0} \frac{i\kappa}{\sqrt{2}n}\,\tilde{\alpha}_n^{25}\left(-\exp(-in\xi^{\mathrm{L}}) + \exp(-in\xi^{\mathrm{R}})\right). \tag{6.19}$$

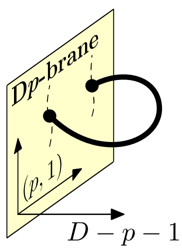
Although the Dirichlet condition $X^{25} = \mathrm{const}.$ may appear unnatural (why X^{25}? what value?), we find that it has to be part of (T-dual) string theory (on compact spaces).

D-Branes. Now we take the Dirichlet condition seriously. At each string boundary we can choose

- Neumann condition $X'^{\mu} = 0$ or
- Dirichlet condition $X^{\mu} = \text{fixed}$.

We can make this choice for each direction μ individually.

Geometrical picture: String ends are confined to so-called "Dp-branes".



(6.20)

- Dp-branes are $(p+1)$-dimensional submanifold of spacetime.
- The submanifold includes the time direction, it has signature $(p,1)$.
- Dirichlet conditions for the $D-p-1$ orthogonal directions of the submanifold.
- Neumann conditions along the remaining $p+1$ directions along the submanifold.
- D-branes can be curved submanifolds.[6]
- Open strings with pure Neumann conditions can be viewed as a spacetime-filling D$(D-1)$-brane.
- T-duality maps between Dp-branes and D$(p\pm 1)$-branes.

Strings propagate on backgrounds with D-branes:

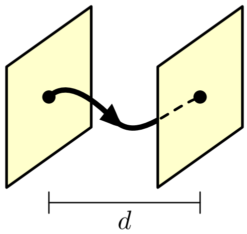
- The bulk spacetime curvature governs the propagation of the string bulk,
- D-branes govern the propagation of string ends.

There is even more to D-branes as non-perturbative objects: We will continue this discussion later.

6.4 Multiple Branes

We can have multiple branes of diverse types. Open strings then stretch between two branes while closed strings are detached from the branes.

Parallel Branes. The simplest case is two parallel planar Dp-branes located at $X^{25}=0,d$ in a non-compact Minkowski space.



(6.21)

The general solution can be sketched as

$$X^\mu = 2\kappa^2 p^\mu \tau + \text{modes}, \quad X^{25} = \frac{\sigma d}{\pi} + \text{modes}. \tag{6.22}$$

The resulting (quantum) mass spectrum from the point of view of $p+1$ dimensions

$$M^2 = \frac{d^2}{4\pi^2\kappa^4} + \frac{1}{\kappa^2}(N-a). \tag{6.23}$$

At the lowest levels we find:

- A scalar particle at level 0 with mass $M=\sqrt{d^2-4\pi^2\kappa^2}/2\pi\kappa^2$.
- A vector particle at level 1 with mass $M=d/2\pi\kappa^2$.
- The scalar particle becomes tachyonic for $d<2\pi\kappa$. Nearby D-branes are unstable.[7]
- The vector particle becomes massless when the branes coincide.[8]

[6] This case is hardly ever treated in practice because it is just as hard as curved spacetimes. Moreover, T-duality would not apply due to absence of a shift symmetry.

[7] This instability can be attributed to the minimum length scale of string theory.

[8] Two D-branes are in fact a stringy formulation of the Higgs mechanism.

Multiple Branes. Consider now N parallel branes located at x_a, $a = 1, \ldots, N$. There are N^2 types of open string (and one type of closed string): The open string vacua distinguished by so-called Chan–Paton factors

$$|0; q; a\bar{b}\rangle_{\mathrm{o}}, \qquad a, b = 1, \ldots, N \tag{6.24}$$

with general mass formula

$$M_{a\bar{b}}^2 = \frac{(x_a - x_b)^2}{4\pi^2\kappa^4} + \frac{1}{\kappa^2}\,(N - a). \tag{6.25}$$

Consider the vector particles at level 1 with mass $|x_a - x_b|/2\pi\kappa^2$. Massless vectors indicate gauge symmetries.

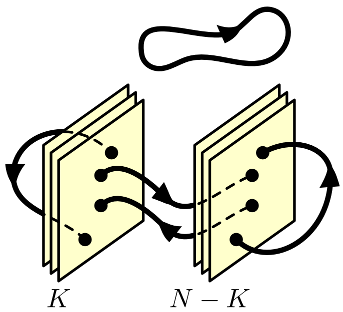
- There are at least N massless vectors. Gauge symmetry: $\mathrm{U}(1)^N$.
- K coincident branes contribute K^2 massless vectors. Enhanced gauge symmetry $\mathrm{U}(1)^K \to \mathrm{U}(K)$.

Massive vectors indicate spontaneously broken symmetries starting from the group $\mathrm{U}(N)$.

Geometric picture of gauge symmetries:

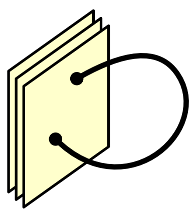
- A stack of N branes has local $\mathrm{U}(N)$ symmetry.
- Separating the branes breaks symmetry to $\mathrm{U}(K) \times \mathrm{U}(N - K)$.
- This makes $2K(N - K)$ of the vectors massive.

$$\frac{\mathrm{U}(N)}{\mathrm{U}(K) \times \mathrm{U}(N-K)} \tag{6.26}$$



One can also represent $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ symmetries: Unoriented strings, strings on orientifolds (spacetime involution paired with orientation reversal).

$$\mathrm{SO}(N), \mathrm{Sp}(N) \tag{6.27}$$



Brane Worlds. We may design many different situations by combining the elements we have encountered so far:

- non-compact dimensions,
- D-branes,
- intersections of D-branes and non-compact dimensions,

- orientifold actions.

It then makes sense to consider the large-scale physics

- along non-compact dimensions,
- within D-branes.

One can investigate the qualitative features of the spectrum:

- Massless vectors indicate gauge symmetries.
- Light vectors indicate spontaneous symmetry breaking.
- Tachyons indicate instabilities of D-branes or spacetime.

In this sense, string theory becomes a framework analogous to QFT which has many degrees of freedom to adjust:

- D-brane arrangements and compact directions (discrete),
- moduli for D-branes and non-compact spaces (continuous).

This is useful for physics: We should try to design the spacetime geometry for string theory such that we obtain the standard model at low energies.

It is also useful for mathematics: String theory involves a lot of (generalised) geometry, and dualities relate various different situations in a non-trivial fashion.

7 Conformal Field Theory

In the previous chapters, we have been discussing the spectrum of the string: starting from the (local) equations of motion and imposing (global) periodicity conditions for either the open or the closed string, we derived the spectrum of the classical string and later for the quantum string. The latter case is described by quantum mechanics for an infinite tower of string modes α_n.

In order to deal with string scattering in the next chapter, we are going to consider the *local* picture on the worldsheet: a quantum field theory for the fields $X(\xi)$, expressed in terms of suitable objects (local operators) on the worldsheet. As pointed out before, due to reparametrisation invariance, the worldsheet coordinates ξ are artificial. Only after fixing the conformal gauge the worldsheet coordinates become meaningful. By fixing the gauge, we have used a good part of the diffeomorphisms: what remains is the (residual) conformal symmetry. Thus we are lead to discuss and explore a quantum field theory based on conformal symmetry: *conformal field theory* (CFT).

As the structure of the final results in a theory is dictated by its symmetries, we will take advantage of conformal symmetry here in order to obtain results efficiently rather than calculating blindly. In particular, conformal symmetry will turn out to be rather constraining and allows to calculate string theory results in a neat and unique way.

While conformal symmetry is the central framework in string theory, it is applicable in versatile situations, for example in many two-dimensional systems in statistical mechanics.

7.1 Conformal Transformations

Conformal transformations are coordinate transformations preserving the angles locally, but allowing for changes in the definition of length. They preserve the metric up to a scale:

$$g'_{\mu'\nu'}(x') = \frac{dx^\mu}{dx'^{\mu'}} \frac{dx^\nu}{dx'^{\nu'}} g_{\mu\nu}(x) \overset{!}{=} f(x)\, g_{\mu'\nu'}(x). \tag{7.1}$$

Action on Coordinates. The following coordinate transformations are comprised by the conformal group:

- Lorentz rotations $x^\mu \to \Lambda^\mu{}_\nu x^\nu$,
- translations $x^\mu \to x^\mu + a^\mu$,
- scale transformations / dila(ta)tions $x^\mu \to s x^\mu$,
- conformal inversions (discrete) $x^\mu \to x^\mu / x^2$,
- conformal boosts (inversion, translation, inversion).

The conformal group in D dimensions is the (universal cover) of $\mathrm{SO}(D,2)$.

Action on Fields. Consider a free scalar with action

$$S \sim \int d^D x \, \tfrac{1}{2} \partial_\mu \Phi(x) \, \partial^\mu \Phi(x). \tag{7.2}$$

In order for the action to be invariant under conformal transformations, the scalar must transform under Lorentz rotations and translations as

$$\Phi'(x) = \Phi(\Lambda x + a). \tag{7.3}$$

Invariance under a scaling $x' = sx$ requires

$$\Phi'(x) = s^{(D-2)/2} \, \Phi(sx), \tag{7.4}$$

while invariance under inversions translates into

$$\Phi'(x) = (x^2)^{-(D-2)/2} \, \Phi(1/x). \tag{7.5}$$

In the same way one can derive similar, but more complicated rules for

- a scalar field $\phi(x)$ with different scaling $\phi'(x) = s^{\Delta} \phi(sx)$,
- spinning fields ρ_μ, ...,
- derivatives $\partial_\mu \Phi$, $\partial_\mu \partial_\nu \Phi$, $\partial^2 \Phi$, ...

2D Conformal Symmetries. In comparison to other dimensions, QFT's in 2D are rather tractable. CFT's in 2D are especially simple. The conformal group splits into

$$\mathrm{SO}(2,2) \simeq \mathrm{SL}(2,\mathbb{R})_{\mathrm{L}} \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{R}}, \tag{7.6}$$

where $\mathrm{SL}(2,\mathbb{R})_{\mathrm{L/R}}$ acts on coordinates as (dropping L/R)

$$\xi' = \frac{a\xi + b}{c\xi + d}, \qquad \delta\xi = \beta + \alpha\xi - \gamma\xi^2. \tag{7.7}$$

Here $\beta^{\mathrm{L/R}}$ are two translations, $\alpha^{\mathrm{L/R}}$ denote rotations and scaling, and $\gamma^{\mathrm{L/R}}$ are two conformal boosts.

The generators of the algebra underlying the group $\mathrm{SL}(2,\mathbb{R})_{\mathrm{L/R}}$ can be supplemented by further generators to yield the infinite-dimensional *Virasoro algebra*. Infinitesimal transformations are then given by

$$\delta\xi^{\mathrm{L/R}} = \epsilon^{\mathrm{L/R}}(\xi^{\mathrm{L/R}}) = \sum\nolimits_n \epsilon_n^{\mathrm{L/R}} (\xi^{\mathrm{L/R}})^{1-n}, \tag{7.8}$$

where the values $n = -1$, $n = 0$ and $n = 1$ correspond to the generators of $\mathrm{SL}(2,\mathbb{R})_{\mathrm{L/R}}$. The boundaries are typically distorted by Virasoro transformations, only a subalgebra preserves them, e.g. $\mathrm{SL}(2,\mathbb{R})$.

7.2 Conformal Correlators

In a quantum theory, one is usually interested in the spectrum of operators, which in our situation is just the string spectrum. Furthermore, one would like to calculate probabilities, which in turn can be derived from expectation values of operators acting on states. In a quantum field theory, there are two formulations available for calculating the vacuum expectation values. Starting from momentum eigenstates, the S-matrix describing the scattering of particles is defined via

$$\langle \vec{q}_1, \vec{q}_2, \ldots | S | \vec{p}_1, \vec{p}_2, \ldots \rangle = \langle 0 | a(\vec{q}_1) a(\vec{q}_2) \ldots S \ldots a^\dagger(\vec{p}_2) a^\dagger(\vec{p}_1) | 0 \rangle. \tag{7.9}$$

In terms of position eigenstates, one calculates time-ordered correlation functions

$$\langle \Phi(x_1) \Phi(x_2) \ldots \rangle = \langle 0 | \mathrm{T}[\Phi(x_1)\Phi(x_2)\ldots] | 0 \rangle. \tag{7.10}$$

Correlator of String Coordinates. As an example of the latter quantity, one can compute a worldsheet correlator using the underlying oscillator relations

$$\begin{aligned} \langle 0 | X^\nu(\xi_2) X^\mu(\xi_1) | 0 \rangle = & -\frac{\kappa^2}{2} \eta^{\mu\nu} \log\bigl(\exp(i\xi_2^{\mathrm{L}}) - \exp(i\xi_1^{\mathrm{L}})\bigr) \\ & - \frac{\kappa^2}{2} \eta^{\mu\nu} \log\bigl(\exp(i\xi_2^{\mathrm{R}}) - \exp(i\xi_1^{\mathrm{R}})\bigr) \\ & + \ldots . \end{aligned} \tag{7.11}$$

Is it possible, to produce the same answer from the worldsheet CFT? In order to answer this question, let us first consider which properties the corresponding CFT correlator should have. Assume a scalar ϕ of dimension Δ and write the correlator

$$\bigl\langle \phi(x_1) \phi(x_2) \bigr\rangle = F(x_1, x_2). \tag{7.12}$$

The correlator should be invariant under translations

$$F(x_1, x_2) = F(x_1 - x_2) =: F(x_{12}), \tag{7.13}$$

which means it should depend on the difference between the two points x_1 and x_2 only. Invariance under Lorentz rotations requires dependence on a Lorentz-scalar

$$F(x_{12}) = F(x_{12}^2)\,. \tag{7.14}$$

Finally, scaling invariance demands the following equality

$$\bigl\langle \phi(x_1)\phi(x_2) \bigr\rangle \stackrel{!}{=} \bigl\langle \phi'(x_1)\phi'(x_2) \bigr\rangle = s^{2\Delta} \bigl\langle \phi(sx_1)\phi(sx_2) \bigr\rangle, \tag{7.15}$$

hence $F(x_{12}^2) = s^{2\Delta} F(s^2 x_{12}^2)$ and

$$F(x_{12}^2) = \frac{N}{(x_{12}^2)^\Delta}\,. \tag{7.16}$$

In summary, there is little freedom left.

Logarithmic Correlator. Remembering the vanishing scaling dimension $\Delta = (D-2)/2 = 0$ of a scalar, we would end up with a constant correlator $F(x_1, x_2) = N$. Fortunately this is not the complete truth: let us take the limit $D = 2 + 2\epsilon$, $N = N_2/\epsilon$ for small ϵ

$$F(x_1, x_2) = \frac{N_2}{\epsilon (x_{12}^2)^\epsilon} \to \frac{N_2}{\epsilon} - N_2 \log x_{12}^2 + \ldots . \tag{7.17}$$

Dropping the leading (divergent) term which is independent of the separation, the correlator can be logarithmic for $\epsilon = 0$ ($\Delta = 0$). Let us now consider the argument in light cone coordinates

$$x_{12}^2 = -x_{12}^{\mathrm{L}} x_{12}^{\mathrm{R}} \tag{7.18}$$

and identify

$$x^{\mathrm{L}} = \exp(i\xi^{\mathrm{L}}), \qquad x^{\mathrm{R}} = \exp(i\xi^{\mathrm{R}}). \tag{7.19}$$

What is the motivation for this identification? It is a two-dimensional conformal transformation which incorporates the closed-string periodicity condition $\sigma \equiv \sigma + 2\pi$ automatically. Of course, one has to choose appropriate coordinates for boundaries in the new coordinates.

So far, almost all conditions on the conformal correlator are taken care of. Only translational invariance is spoiled: the string coordinates are functions of $x^{\mathrm{L/R}}$ except for the linear dependence on $\tau = -\frac{i}{2}\log(x^{\mathrm{L}} x^{\mathrm{R}})$. The solution to this problem is obvious: instead of choosing the field X alone, take $\partial X^\mu / \partial x^{\mathrm{L/R}}$:

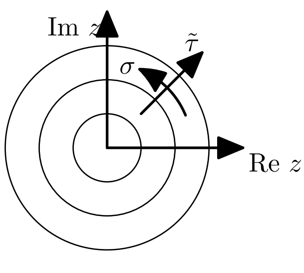
$$\langle 0 | \partial_{\mathrm{L}} X^\nu(x_2) \partial_{\mathrm{L}} X^\mu(x_1) | 0 \rangle = \frac{-\frac{1}{2}\kappa^2 \eta^{\mu\nu}}{\left(x_2^{\mathrm{L}} - x_1^{\mathrm{L}}\right)^2} . \tag{7.20}$$

This is indeed closer to the expected form of a conformal correlator!

Wick Rotation. In the context of conformal field theory, the worldsheet is conventionally taken to have euclidean signature. In order to get there, perform the Wick rotation $\tau = -i\tilde{\tau}$ (now $\tilde{\tau}$ is real) and define

$$\exp(i\xi^{\mathrm{L}}) = \exp(\tilde{\tau} - i\sigma) =: \bar{z}, \quad \exp(i\xi^{\mathrm{R}}) = \exp(\tilde{\tau} + i\sigma) =: z. \tag{7.21}$$

The result of this change are *cylindrical coordinates* for the (euclidean) string:



(7.22)

The radius $|z|$ denotes the exponential euclidean time $\tilde{\tau}$, while σ is mapped to the (naturally periodic) angular coordinate. These coordinates are the standard ones for a euclidean quantum field theory: the worldsheet coordinates z and $\bar{z}$ are complex conjugates and the fields are functions $f(z,\bar z)$ of complex z. Splitting of the string coordinates into a right and a left part is replaced by considering a holomorphic and a anti-holomorphic part:

$$X(z,\bar z) = X(z) + \bar X(\bar z). \tag{7.23}$$

Why is it favourable to use this language? Firstly, conformal transformations preserve holomorphicity, that is, they do not mix the functions $X(z)$ and $\bar X(\bar z)$. Secondly (and most importantly) one can now employ the powerful tools of complex functional analysis, e.g. residue theorems.

7.3 Local Operators

In the last subsection we argued that one should use the derivative $\partial X(z)$ and $\bar\partial \bar X(\bar z)$ of the string coordinate field $X(z,\bar z) = X(z) + \bar X(\bar z)$ in order to obtain a quantity without linear dependence on τ.

The basic objects in a CFT are local operators $\mathcal{O}_i(z,\bar z)$, which are built from products of fields X and their derivatives $\partial^n\bar\partial^{\bar n}X$, all of them evaluated at the same point $(z,\bar z)$ on the worldsheet. The normal ordering $\mathcal{O}_i = {:}\ldots{:}$ is implicit for local operators, thus there are no self-correlations. Typical examples are $\mathcal{O}_1 = {:}(\partial X)^2{:}$ and $\mathcal{O}_2^{\mu\nu} = {:}X^\mu\partial X^\nu{:} - {:}X^\nu \partial X^\mu{:}$.

In a classical setup, local operators behave as the "sum of their constituents", while quantum-mechanically, they need to be considered as inedependent quantities: recall for example the quantum effects for the Virasoro charges $(\partial X)^2$! In the remainder of this subsection we will classify local operators in a CFT. The classification is based upon their behaviour under conformal transformations.

Descendants. Under shifts $(\delta z, \delta\bar z) = (\epsilon, \bar\epsilon)$, all local operators transform as

$$\begin{aligned} \mathcal{O}(z+\epsilon, \bar z + \bar\epsilon) &= \mathcal{O}(z,\bar z) + \epsilon\partial\mathcal{O} + \bar\epsilon\bar\partial\mathcal{O} \\ \delta\mathcal{O} &= \epsilon\partial\mathcal{O} + \bar\epsilon\bar\partial\mathcal{O}. \end{aligned} \tag{7.24}$$

An operator $\partial^n\bar\partial^{\bar n}\mathcal{O}$ is called a descendant of $\mathcal{O}$. Descendants are of little importance in conformal field theory, as their structure and properties are completely known from the operator $\mathcal{O}$. Alternatively, one can argue that shifts are symmetries and thus there is no need to consider descendants.

Weights. Most local operators (notable exception: X) are classified by their weights $(h, \bar h)$. Considering a scaling/rotation, that is $(z,\bar z) \to (sz, \bar s\bar z)$ or $\delta(z,\bar z) = (\epsilon z, \bar\epsilon \bar z)$ one finds

$$\begin{aligned} \mathcal{O}'(z,\bar z) &= s^h \bar s^{\bar h}\mathcal{O}(sz, \bar s\bar z), \\ \delta\mathcal{O} &= \epsilon(h\mathcal{O} + z\partial\mathcal{O}) + \bar\epsilon(\bar h\mathcal{O} + \bar z\bar\partial\mathcal{O}). \end{aligned} \tag{7.25}$$

In reference to the type of transformation, scaling and rotation, $\Delta = h + \bar{h}$ is called the *scaling dimension* where $S = h - \bar{h}$ is the *conformal spin*.

For a unitary CFT, both h and $\bar{h}$ have to be real and non-negative. Here are two examples of weights for some fields: $\partial X \to (1, 0)$ and $(\partial X)^2 \to (2, 0)$. The field X does not have proper weights.

In a classical theory, the products of local operators $\mathcal{O} = \mathcal{O}_1\mathcal{O}_2$ will have a weight equal to the sum of individual weights. In a quantum theory, weights are usually not additive, due to implicit normal ordering.

Quasi-Primary Operators. A local operator with weights $(h, \bar{h})$ is called quasi-primary if

$$\mathcal{O}'(z, \bar{z}) = \left(\frac{dz'}{dz}\right)^h \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{\bar{h}} \mathcal{O}(z', \bar{z}') \tag{7.26}$$

for all $\mathrm{SL}(2, \mathbb{C})$ Möbius transformations

$$z' = \frac{az + b}{cz + d}, \qquad \bar{z}' = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}. \tag{7.27}$$

For infinitesimal boosts $\delta(z, \bar{z}) = (\epsilon z^2, \bar{\epsilon}\bar{z}^2)$ it must satisfy

$$\delta\mathcal{O} = \epsilon(2hz\mathcal{O} + z^2\partial\mathcal{O}) + \bar{\epsilon}(2\bar{h}\bar{z}\mathcal{O} + \bar{z}^2\bar{\partial}\mathcal{O}). \tag{7.28}$$

Descendants of quasi-primary operators are not quasi-primary. In a CFT it is sufficient to deal with quasi-primary operators only.

Primary Operators. Primary operators are a subclass of the quasi-primary operators described above. An operator is called primary, if it satisfies the quasi-primary conditions for *all* transformations

$$(z, \bar{z}) \to (z'(z), \bar{z}'(\bar{z})) \quad \text{or} \quad (\delta z, \delta\bar{z}) = (\zeta(z), \bar{\zeta}(\bar{z})). \tag{7.29}$$

Infinitesimally, one finds

$$\delta\mathcal{O} = (h\,\partial\zeta\,\mathcal{O} + \zeta\,\partial\mathcal{O}) + (\bar{h}\,\bar{\partial}\bar{\zeta}\,\mathcal{O} + \bar{\zeta}\,\bar{\partial}\mathcal{O}). \tag{7.30}$$

Of course, the infinitesimal transformations above can be derived from the general rule here easily.

Correlators are locally conformal invariant. Only a subclass of transformations (e.g. Möbius) leaves the correlator globally invariant. As an example, consider

$$\langle \mathcal{O}_1^\mu \mathcal{O}_2^\nu \rangle = \frac{-\frac{1}{2}\kappa^2\eta^{\mu\nu}}{(z_1 - z_2)^2}\,. \tag{7.31}$$

built from the primary operator $\mathcal{O}^\mu = \partial X^\mu$ with weights $(h, \bar{h}) = (1, 0)$. The correlator is exactly invariant under the transformation $\delta z = z^{1-n}$, if $|n| \le 1$ (Möbius transformations), but invariant up to polynomials for $|n| > 1$.[1]

[1] The polynomial contributions are small w.r.t. the pole $1/(z_1 - z_2)^2$ and can therefore be ignored to some extent.

State-Operator Map. There is a one-to-one map between

- quantum states on a cylinder $\mathbb{R} \times S^1$ and
- local operators (at $z = 0$).

Below, it will be convenient to use variables obtained by the conformal map

$$z = \exp(+i\zeta), \quad \bar{z} = \exp(-i\bar{\zeta}), \tag{7.32}$$

where

$$\zeta, \bar{\zeta} = \sigma \mp i\tilde{\tau}. \tag{7.33}$$

A state is then given by a wave function at constant $\tilde{\tau} = -\operatorname{Im}\zeta$.

- Time evolution is equivalent to radial evolution in the z-plane.
- Asymptotic time $\tilde{\tau} \to -\infty$ corresponds to $z = 0$.
- Local operator at $z = 0$ is used to excite asymptotic wave function.
- The unit operator 1 corresponds to the vacuum.

7.4 Operator Product Expansion

In a CFT we wish to compute correlation functions

$$\big\langle \mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2)\ldots\mathcal{O}_n(\xi_n)\big\rangle = F_{12\ldots n}. \tag{7.34}$$

Suppose $\xi_1 \approx \xi_2$; then one can Taylor expand

$$\mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2) = \sum_{n=0}^{\infty} \frac{1}{n!}(\xi_2 - \xi_1)^n \mathcal{O}_1(\xi_1)\partial^n \mathcal{O}_2(\xi_1). \tag{7.35}$$

The expansion converts local operators at two points into a sum of local operators at a single point. The classical statement is exact.

Quantum OPE. Quantum-mechanically there are additional contributions from operator ordering because normal ordering is implicit. However, the product of local operators can still be written as sum of some local operators

$$\mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2) = \sum\nolimits_i C^i_{12}(\xi_2 - \xi_1)\mathcal{O}_i(\xi_1). \tag{7.36}$$

The above statement has to be understood in terms of a correlator. More precisely, with the insertion of any (non-local combination of) operators "…" it reads

$$\big\langle \mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2)\ldots\big\rangle = \sum\nolimits_i C^i_{12}(\xi_2 - \xi_1)\big\langle \mathcal{O}_i(\xi_1)\ldots\big\rangle. \tag{7.37}$$

This statement is called *operator product expansion* (OPE), where $C^k_{ij}(\xi_2 - \xi_1)$ are called structure constants and conformal blocks. The sum extends over all local operators (including descendants).

This is a very powerful idea: every (non-local) operator can be written as an expansion in local operators. This statement is analogous to the multipole expansion of electrodynamics. The formalism works exactly in any CFT and is a central tool.

Higher Points. Formally, one can compute higher-point correlation functions:

$$F_{123\ldots n} = \sum\nolimits_i c_{12}^i F_{i3\ldots n}. \qquad (7.38)$$

Recursively, one can reduce to the one-point function, which is trivial except for the unit operator 1

$$\langle \mathcal{O}_i \rangle = 0, \quad \langle 1 \rangle = 1. \qquad (7.39)$$

Higher-point functions can thus be reduced to a sequence of structure constants C_{ij}^k. This is a vast simplification: one needs only C_{ij}^k in order to describe correlators in CFT. In practice, the structure constants are hard to compute and moreover the result of a single OPE are infinitely many local operators. Superficially, the result seems to depend on the sequence of reducing correlators using OPE's. This is of course not true because the structure constants are special quantities which obey crossing relations that ensure independence of the decomposition.

Lower Points. The two-point function is equivalent to an OPE onto the unit operator

$$F_{ij} = \langle \mathcal{O}_i \mathcal{O}_j \rangle = \sum\nolimits_k C_{ij}^k \langle \mathcal{O}_k \rangle = C_{ij}^1. \qquad (7.40)$$

The OPE constants are determined by the three-point functions

$$F_{ijk} = \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = \sum\nolimits_l C_{ij}^l \langle \mathcal{O}_k \mathcal{O}_l \rangle = \sum\nolimits_l F_{kl} C_{ij}^l. \qquad (7.41)$$

Lower-point functions are restricted tightly by conformal symmetry:

- Two-point function are non-trivial only for operators related by conformal symmetry (descendants).
- Non-trivial three-point functions exist for three unrelated operators.

Conformal symmetry allows to map a triple of points to any other triple of points. Consequently, there are no conformally invariant functions depending on only two or three points. This implies the following for correlation functions:

- The coordinate dependence of a two-point function is fixed

$$F_{ij} \sim \frac{N_{ij}}{|\xi_i - \xi_j|^{2\mathit{\Delta}_i}}\,. \qquad (7.42)$$

- The coordinate dependence of a three-point function is fixed

$$F_{ijk} \sim \frac{N_{ijk}}{|\xi_i - \xi_j|^{\mathit{\Delta}_{ij}} |\xi_j - \xi_k|^{\mathit{\Delta}_{jk}} |\xi_k - \xi_i|^{\mathit{\Delta}_{ki}}} \qquad (7.43)$$

  with scaling weights $\mathit{\Delta}_{ij} = \mathit{\Delta}_i + \mathit{\Delta}_j - \mathit{\Delta}_k$. The numerators N_{ij} and N_{ijk} depend on dimension, spin, level of descendant and operator normalisation.
- Higher-point functions can depend on the available conformally invariant cross ratios in an arbitrary fashion.

Once one has chosen a normalisation for operators, the data in a CFT consists of

- scaling dimensions along with the spins of the local operators: spectrum,
- coefficients of three-point function: structure constants.

7.5 Stress-Energy Tensor

The Noether currents for spacetime symmetries are encoded in the conserved stress-energy tensor $T_{\alpha\beta}$

$$T_{\alpha\beta} = -\frac{1}{4\pi\kappa^2}\big((\partial_\alpha X)\cdot(\partial_\beta X) - \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}(\partial_\gamma X)\cdot(\partial_\delta X)\big)\,. \tag{7.44}$$

This is an object of central importance for CFT and for the OPE! We know that the trace is exactly zero because of Weyl symmetry. The two remaining components T_{LL} and T_{RR} translate into the euclidean quantities

$$T = -\frac{1}{\kappa^2}(\partial X)^2, \qquad \bar{T} = -\frac{1}{\kappa^2}(\bar{\partial}\bar{X})^2. \tag{7.45}$$

Let us ignore the string physical state condition which eventually sets them to zero, $T = \bar{T} = 0$.

Conservation. The current $J(z) = \zeta(z)T(z)$ corresponds to the infinitesimal transformation $\delta z = \zeta(z)$. While classical conservation $\bar{\partial}J = 0$ is ensured by means of the e.o.m., in quantum mechanics the conservation law is implied by the Ward identity:

$$\bar{\partial}J(z)\mathcal{O}(w,\bar{w}) = 2\pi\,\delta^2(z-w,\bar{z}-\bar{w})\,\delta\mathcal{O}(w,\bar{w}). \tag{7.46}$$

Thus, the current $J(z)$ is conserved everywhere except at the operator locations. In order to calculate the OPE, one integrates z over a small ball around w:

$$\frac{1}{2\pi}\int_{|z-w|<\epsilon} d^2z\ \ldots\,. \tag{7.47}$$

In a first step, let us evaluate the integration over $\bar{z}$ ($\int d^2z\bar{\partial}\ldots = -i\int dz\ldots$) and obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_{|z-w|=\epsilon} dz\, J(z)\mathcal{O}(w,\bar{w})\\ &= \frac{1}{2\pi i}\int_{|z-w|=\epsilon} dz\,\zeta(z)T(z)\mathcal{O}(w,\bar{w})\\ &= \delta\mathcal{O}(w,\bar{w}).\end{aligned} \tag{7.48}$$

An analogous calculation can be done for $\bar{T}$. In the following, we will consider the holomorphic part only.

Stress-Energy OPE. In the last paragraph of this chapter, we would like to derive the OPE of an operator $\mathcal{O}$ and the stress-energy tensor T. So let us consider the previous equation for several types of infinitesimal transformations.

First consider the translation $\delta z = \epsilon$, $\delta\mathcal{O} = \epsilon\partial\mathcal{O}$. In order to generate the residue, we need a simple pole:

$$T(z)\mathcal{O}(w,\bar{w}) = \ldots + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots\,. \tag{7.49}$$

Further terms with higher poles and polynomials in "…" are unconstrained from considering just infinitesimal translations. So let us start with an operator $\mathcal{O}$ of holomorphic weight h and consider a scaling $\delta z = \epsilon z$, $\delta\mathcal{O} = \epsilon(h\mathcal{O} + z\partial\mathcal{O})$. Upon substitution, one needs to require the following poles in the OPE:

$$T(z)\mathcal{O}(w,\bar{w}) = \ldots + \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots . \tag{7.50}$$

In a next step, suppose $\mathcal{O}$ is quasi-primary. Considering the scaling $\delta z = \epsilon z^2$, $\delta\mathcal{O} = \epsilon(2hz\mathcal{O} + z^2\partial\mathcal{O})$ and substituting, we need to require the absence of the cubic pole

$$T(z)\mathcal{O}(w,\bar{w}) = \ldots + \frac{0}{(z-w)^3} + \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots . \tag{7.51}$$

Finally suppose $\mathcal{O}$ is primary, which is equivalent to demanding the absence of higher poles

$$T(z)\mathcal{O}(w,\bar{w}) = \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots . \tag{7.52}$$

When there is a derivative acting on an operator, the poles are shifted by one order. Thus, descendants are not (quasi-)primary operators.

Let us state the OPE of the stress-energy tensor. An explicit computation using Wick's theorem yields:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots . \tag{7.53}$$

This is a general result in CFT's. It encodes the Virasoro algebra! Let us note a couple of properties of the stress-energy tensor:

- T is a local operator,
- T has holomorphic weight $h = 2$ (classically),
- T is quasi-primary,
- T is not primary (unless $c = 0$),
- the quartic pole carries central charge $c = D$.

Conformal transformations for the stress-energy tensor T are almost primary:

$$\begin{aligned}
\delta T &= \delta z\,\partial T + 2\,\partial\delta z\,T + \frac{c}{12}\,\partial^3\delta z,\\
T'(z) &= \left(\frac{dz'}{dz}\right)^2 \left(T(z') + \frac{c}{12}\,S(z',z)\right),\\
S(z',z) &= \left(\frac{d^3z'}{dz^3}\right)\left(\frac{dz'}{dz}\right)^{-1} - \frac{3}{2}\left(\frac{d^2z'}{dz^2}\right)^2\left(\frac{dz'}{dz}\right)^{-2} .
\end{aligned} \tag{7.54}$$

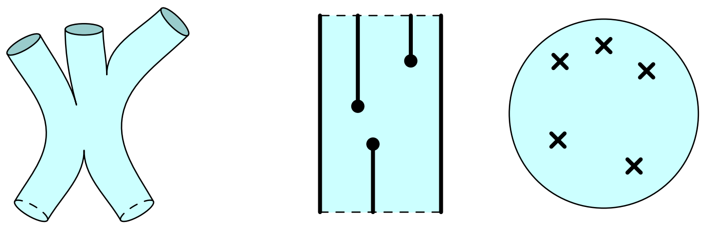
The additional term S is the *Schwarzian derivative*, which vanishes for Möbius transformations.

8 String Scattering

To obtain a basic understanding of string interactions we shall compute some scattering amplitudes. As with conventional particles, prepare an initial state containing several strings,[1] make them collide and produce several outgoing strings.

There are two fundamental approaches:

- Minkowskian/QFT picture: Consider a single string worldsheet with coordinates τ, σ where σ is constrained to some finite range as before. Cuts the worldsheet at specific values of σ and for some range of τ. At these cuts, the boundary conditions are altered. Finally, integrate over all configurations of additional boundaries.
- Euclidean/CFT picture: Insert vertex operators into the worldsheet. Each vertex corresponds to an asymptotic string via an exponential map. Integrate over the locations of the punctures.



(8.1)

8.1 Vertex Operators

We need to find a dictionary between string states and vertex operators, the so-called state-operator map:

- Which operator creates a string?
- How to specify the momentum q?
- How to specify the string modes?

The solution turns out to be based on the operator

$$\mathcal{O}[q] = {:}\exp(iq_\mu X^\mu){:}. \tag{8.2}$$

Why? This operator is a momentum eigenstate, it receives a phase $\exp(iq_\mu\epsilon^\mu)$ for the translation $X \to X + \epsilon$

$$\mathcal{O}[q] \mapsto \exp(iq_\mu\epsilon^\mu)\mathcal{O}[q]. \tag{8.3}$$

[1]Each piece of string carries its own string vacuum $|0; q\rangle$ with individual momenta and string excitations.

String Vacuum. Let us investigate the operator further. In the CFT picture, compute the OPE with the stress-energy tensor T

$$T(z)\mathcal{O}[q](w,\bar{w}) = \frac{\frac{1}{4}\kappa^2 q^2 \mathcal{O}[q](w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}[q](w,\bar{w})}{z-w} + \ldots . \tag{8.4}$$

It is a primary operator with weights $(\frac{1}{4}\kappa^2 q^2, \frac{1}{4}\kappa^2 q^2)$!

- Note that X itself is not primary and therefore putting X in the exponent makes an actual difference!
- We obtain a non-trivial, non-integer weight.
- The weight is a quantum effect $\sim \kappa^2$.

Consider the two-point correlator

$$\langle \mathcal{O}_1[q_1]\mathcal{O}_2[q_2] \rangle \simeq |z_1 - z_2|^{\kappa^2(q_1\cdot q_2)}. \tag{8.5}$$

In this calculation, the zero mode $X^\mu = x^\mu + \ldots$ contributes an extra factor which is not obvious at first sight, but required for consistency

$$\int d^D x \exp(iq_1 \cdot x + iq_2 \cdot x) \sim \delta^D(q_1 + q_2). \tag{8.6}$$

Hence the two-point function is compatible with the primary property of weight $(\frac{1}{4}\kappa^2 q^2, \frac{1}{4}\kappa^2 q^2)$

$$\langle \mathcal{O}_1[q_1]\mathcal{O}_2[q_2] \rangle \simeq \frac{\delta^D(q_1+q_2)}{|z_1 - z_2|^{\kappa^2 q_1^2}} . \tag{8.7}$$

The operator $\mathcal{O}[q](z,\bar{z})$ creates a string state at the worldsheet location $(z,\bar{z})$. The worldsheet location is unphysical, hence integrate over all potential insertion points:

$$V[q] = g_\mathrm{s} \int d^2 z\, \mathcal{O}[q](z,\bar{z}). \tag{8.8}$$

Importantly, one can only integrate weight-$(1,1)$ primary operators.[2] Hence:

- Mass is determined $M^2 = -q^2 = -4/\kappa^2$; The state describes the string tachyon!
- Intercept $a = \bar{a} = 1$ determined by worldsheet integration.

Excited Strings. What about excited strings? A level-1 state corresponds to

$$V^{\mu\nu}[q] = g_\mathrm{s} \int d^2 z\, \partial X^\mu \bar{\partial} X^\nu \mathcal{O}[q]. \tag{8.9}$$

- Weight is $(1 + \frac{1}{4}\kappa^2 q^2, 1 + \frac{1}{4}\kappa^2 q^2) = (1,1)$ for massless q.
- Primary condition removes unphysical polarisations, e.g.

$$T(z)\,\partial X^\mu(w)\mathcal{O}[q](w,\bar{w}) \sim \frac{q^\mu \mathcal{O}[q]}{(z-w)^3} + \ldots . \tag{8.10}$$

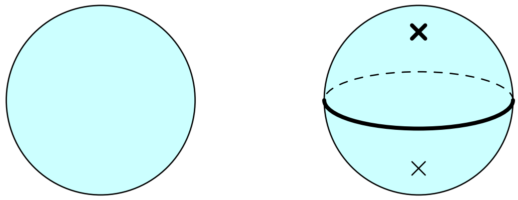
[2] A scaling transformation of the integration variables is compensated by a scaling transformation of the primary operator. Unless the two resulting factors compensate, the integral must be zero or ill-defined.

- Gauge d.o.f. are total derivatives

$$q_\nu V^{\mu\nu}[q] = -ig_\mathrm{s} \int d^2z\, \bar\partial\big(\partial X^\mu \mathcal{O}[q]\big) = 0. \tag{8.11}$$

We obtain the following vertex operator picture:

- CFT vacuum is empty worldsheet (genus 0, no punctures).
- $\int d^2z\, \mathcal{O}[q](z,\bar z)$ is string vacuum $|0;q\rangle$ (add puncture).
- $\int d^2z\, \ldots\, \mathcal{O}[q](z,\bar z)$ are excited string states.
- Insertions of $\partial^n X^\mu$ correspond to string oscillators α^μ_n; insertions of $\bar\partial^n X^\mu$ correspond to $\bar\alpha^\mu_n$.



(8.12)

8.2 Veneziano Amplitude

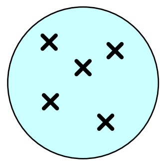
Consider an n-point amplitude (with $\mathcal{O}_k = \mathcal{O}[q_k](z_k,\bar z_k)$)

$$A_n \sim \frac{1}{g_\mathrm{s}^2}\langle V_1 \ldots V_n\rangle \sim g_\mathrm{s}^{n-2} \int d^{2n}z \langle \mathcal{O}_1 \ldots \mathcal{O}_n\rangle. \tag{8.13}$$

It is simplest to use tachyon vertex operators. We could do others, but it adds complication (fields). The computation and the result are qualitatively the same.

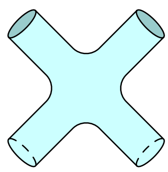
Perform Wick contractions and zero mode integration to obtain

$$\big\langle \mathcal{O}_1 \ldots \mathcal{O}_n \big\rangle \sim \delta^D(Q) \prod\nolimits_{j<k} |z_j - z_k|^{\kappa^2 q_j\cdot q_k}. \tag{8.14}$$



The integral is invariant under Möbius transformations (note that $q_k^2 = 4/\kappa^2$). We map three punctures to fixed positions $z_1 = \infty$, $z_2 = 0$, $z_3 = 1$.[3] The remaining integral for $n = 4$ external strings reads

$$A_4 \sim g_\mathrm{s}^2 \delta^D(Q) \int d^2z\, |z|^{\kappa^2 q_2\cdot q_4} |1-z|^{\kappa^2 q_3\cdot q_4}. \tag{8.15}$$



It can be performed exactly and yields a combination of gamma functions

$$A_4 \sim g_\mathrm{s}^2 \delta^D(Q) \frac{\Gamma(-1-\kappa^2 s/4)\,\Gamma(-1-\kappa^2 t/4)\,\Gamma(-1-\kappa^2 u/4)}{\Gamma(+2+\kappa^2 s/4)\,\Gamma(+2+\kappa^2 t/4)\,\Gamma(+2+\kappa^2 u/4)}\,. \tag{8.16}$$

[3] In fact, this transformation amounts to a factor of the divergent integral $\int d^2z_1\, d^2z_2\, d^2z_3$. This integral does not depend on any external data and should be factored out from any amplitude calculation. The 6 integrations correspond to the 6 global conformal symmetries.

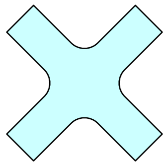
For convenience, we have introduced the Mandelstam invariants s, t, u

$$s = -(q_1 + q_2)^2, \quad t = -(q_1 + q_4)^2, \quad u = -(q_1 + q_3)^2, \tag{8.17}$$

with the relation $s + t + u = -q_1^2 - q_2^2 - q_3^2 - q_4^2 = -16/\kappa^2$.

This is the Virasoro-Shapiro amplitude for closed strings. The corresponding amplitude for open strings reads

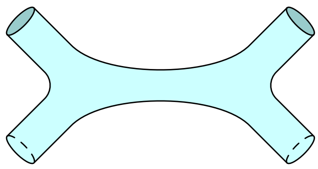
$$A_4 \sim g_\mathrm{s} \frac{\Gamma(-1 - \kappa^2 s)\,\Gamma(-1 - \kappa^2 t)}{\Gamma(+2 + \kappa^2 u)}\,. \tag{8.18}$$



It was proposed (not calculated) earlier by Veneziano based on crossing symmetry. This is considered as the birth of string theory (as the so-called dual resonance model).

Amplitudes have many desirable features:

- Poles at $s, t, u = (N-1)4/\kappa^2$ or $s, t = (N-1)/\kappa^2$ correspond to an infinite tower of virtual particles exchanged. The mass spectrum coincides with closed and open strings.



(8.19)

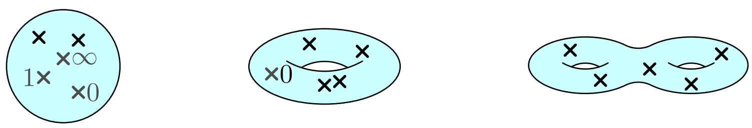
- Residues indicate spin $J = 2N$ or $J = N$. Regge trajectory!
- Soft behaviour at $s \to \infty$. Even for gravitons!
- Manifest crossing symmetry $s \leftrightarrow t \leftrightarrow u$ or $s \leftrightarrow t$. Amazing!

Not possible to obtain such a result from QFT with finitely many particles.

8.3 String Loops

The result is exact as far as α' is concerned. Strings on flat space is a free theory in α'!

However, worldsheet topology matters. String loop corrections correspond to adding handles to the surface: higher genus. The power of g_s reflects the Euler characteristic of the worldsheet. It turns out that at string scattering at genus h requires $2n + 6h - 6$ integrations



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Tree Level. At tree level the worldsheet is a sphere or a disk with n punctures. Euler characteristic is $-2 + n$ or $-1 + n/2$, respectively. Due to 6 global conformal symmetries, the pertinent integration is over $n - 3$ points, cf. above.

One Loop. At one loop the worldsheet is a torus with n punctures. Its Euler characteristic is n. The torus has 2 moduli and 2 shifts as global transformations. Therefore the integration is over 2D Teichmüller space and over $n-1$ external vertices which amounts to $2n$ integrations. The result is expressed using elliptic and modular functions. Feasible problem!

Two Loops. At two loops the worldsheet is a 2-torus with n punctures and Euler characteristic $2+n$. This torus has 6 moduli, but no global shifts: Therefore we have to perform $2(n+3)$ integrations. This is very hard, but can sometimes be done. Higher-loop results are typically inaccessible.

9 General Relativity Basics

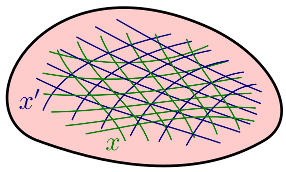
This chapter is a very brief introduction to general relativity. We shall focus on the topics that are needed for this course of string theory, namely the definition of the metric tensor, the Einstein equations, gravitational waves and graviton scattering.

9.1 Differential Geometry

Ordinary field theories are formulated on a flat Minkowski space. General relativity is a theory of the curvature of spacetime. We therefore have to introduce a framework to deal with curved spaces and spacetimes. This is Riemannian geometry which in turn is based on differential geometry. Differential geometry describes the basic notion of fields, coordinates and vector spaces on a general differentiable manifold $\mathcal{M}$. Riemannian geometry equips manifolds with the concept of metric to measure distances and angles which can carry curvature.[1]

Diffeomorphisms. We are used to express the objects in field theory such as fields and derivatives in terms of coordinates. In flat Minkowski geometry, there is a small set of preferred coordinate systems, the inertial frames. Calculus in these frames is straightforward and different frames are related by Poincaré transformations. Special relativity essentially is the statement that laws of nature should be the same in every reference frame.

In a general curved spacetime $\mathcal{M}$, there is no preferred set of coordinates. In analogy to special relativity one may therefore demand that the laws of nature should be independent of the choice of coordinates. This is an axiom of general relativity. This implies that general changes of coordinates, so-called *diffeomorphisms*, must be symmetries of the theory.[2]



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For a scalar field $F(x)$, a diffeomorphism $x \mapsto x'$ maps the field $F \mapsto F'$ such that

$$F'(x') = F(x). \tag{9.2}$$

[1]Note that the curved space should be viewed as an intrinsic space, not (necessarily) embedded into some higher-dimensional space.

[2]This statement is to some extent empty because *every theory* can be formulated in arbitrary coordinate systems (e.g. spherical coordinates, cylindrical coordinates) such that appropriately defined equations are invariant under diffeomorphisms. A more accurate statement for general relativity is that the diffeomorphism-invariant equations take a simple and natural form. Unfortunately, this statement is just an aesthetic one.

The transformation rules for partial derivatives and vector fields require some more care. However, concatenation of diffeomorphisms $x \mapsto x' \mapsto x''$ must act transitively, $x \mapsto x''$.

Vectors, Covectors, Tensors. Let us next understand how partial derivatives and vector fields transform under diffeomorphisms.

Define G_μ as the partial derivative ∂_μ of a (scalar) function F

$$G_\mu(x) := \frac{\partial}{\partial x^\mu} F(x). \tag{9.3}$$

We know how the derivative transforms under coordinate transformations

$$G'_{\mu'}(x') := \frac{\partial}{\partial x'^{\mu'}} F'(x') = \frac{\partial x^\nu}{\partial x'^{\mu'}} \frac{\partial}{\partial x^\nu} F(x) = \frac{\partial x^\nu}{\partial x'^{\mu'}} G_\nu(x). \tag{9.4}$$

In addition to the change of coordinates, the basis of the vector space is transformed by the linear map $\partial x^\nu / \partial x'^{\mu'}$. Note that this rule is compatible with the transformation rule for vectors under Poincaré transformations

$$x'^\mu = A^\mu{}_\nu x^\nu + B^\mu, \qquad \frac{\partial x^\nu}{\partial x'^{\mu'}} = (A^{-1})^\nu{}_{\mu'}. \tag{9.5}$$

Now we can generalise the above transformation rule for covariant derivatives to a new class of fields: A *covector field* $F_\mu(x)$ is a field which transforms according to

$$F'_{\mu'}(x') = \frac{\partial x^\nu}{\partial x'^{\mu'}} F_\nu(x). \tag{9.6}$$

A vector field F^μ is the dual of a covector field G_μ. The contraction of the two should be a scalar field which transforms appropriately

$$F'^{\mu'}(x') G'_{\mu'}(x') = F^\mu(x) G_\mu(x). \tag{9.7}$$

The vector index therefore transforms with the inverse transformation matrix

$$F'^{\mu'}(x') = \frac{\partial x'^{\mu'}}{\partial x^\nu} F^\nu(x). \tag{9.8}$$

In what follows we will have to combine the concept of vector and covector fields in so-called *tensor fields.* A tensor field of rank (a, b) transforms as the tensor product of a vector fields and b covector fields, e.g.

$$F'^{\mu'\nu'}{}_{\rho'}(x') = \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial x'^{\nu'}}{\partial x^\nu} \frac{\partial x^\rho}{\partial x'^{\rho'}} F^{\mu\nu}{}_\rho(x). \tag{9.9}$$

The above definition of tensor fields is based on partial derivatives of scalar fields. Note, however, that the partial derivatives of generic tensor fields are not tensor fields. This is because, in diffeomorphisms, the covariant derivatives can act on the transformation matrix for the tensor indices spoiling the tensor character of the field. The deeper reason for this behaviour is that the tangent spaces at different points are a priori unrelated. The notable exception are among anti-symmetric covector indices, so called differential forms. For example, for any covector field F_μ, the field

$$G_{\mu\nu} := \partial_\mu F_\nu - \partial_\nu F_\mu \tag{9.10}$$

transforms as a tensor of rank $(0, 2)$.

Index-Free Formulation. The notion of indices of tensor fields is tightly bound to the respective coordinate systems. It is sometimes easier to deal with index-free quantities, which is also closer to the mathematical literature. As an aside, let us introduce these.

An index-free vector field F is defined as a derivative operator

$$F(x) = F^\mu(x) \frac{\partial}{\partial x^\mu} \,. \tag{9.11}$$

It is evident that such derivative operators define a vector space. With the above transformation rule of a vector field F^μ with indices, F transforms as a (derivative-valued) scalar field. The partial derivatives $\partial/\partial x^\mu$ serve as a basis of vector fields,[3] and the F^μ are the basis coefficients.

An index-free covector field F is defined as the *differential one-form*

$$F(x) = F_\mu(x)\, dx^\mu. \tag{9.12}$$

The one-forms dx^μ serve as a basis of differential one-forms and the covector field F_μ with indices as the basis coefficients.

Differential one-forms are the duals of the partial derivative operators

$$dx^\mu(\partial x_\nu) = \delta^\mu_\nu. \tag{9.13}$$

The differential of a scalar field dF is defined as the covector field

$$dF = dx^\mu\, \partial_\mu F. \tag{9.14}$$

It satisfies the property that for any vector field X

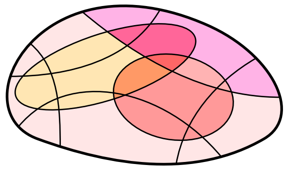
$$XF = X^\mu \partial_\mu F = X^\mu \partial_\nu F\, dx^\nu(\partial_\mu) = dF(X). \tag{9.15}$$

Atlases and Bundles. Let us add some general considerations of differential geometry which may not be immediately relevant to our purposes.

In differential geometry, coordinates are hardly ever globally defined on a manifold. They are usually only valid in local (open) patch of the manifold as a so-called *chart*. A manifold is covered by a set of overlapping charts, a so-called

[3] Somewhat confusingly, $\partial/\partial x^\mu$ obeys the transformation rules of a covector field ∂_μ with indices. However, this is implied by the requirement that the index-free vector field transforms trivially.

atlas. In the overlap of two charts, the coordinates are translated by the *transition map* which must be diffeomorphism. Whenever three charts overlap, the diffeomorphisms must be compatible.



(9.16)

Charts and transition maps provide a consistent mathematical definition of the physicists' notion of coordinates and diffeomorphisms. Importantly, these are local concepts on the manifold. In fact, for many manifolds there is no global chart which covers the whole manifold. In other words, any given coordinate system may not describe the whole universe. Similarly, diffeomorphisms need not cover the whole coordinate system.

An atlas may contain any number of charts such that the manifold is covered multiple times. One can also introduce additional charts to accommodate for other choices of coordinate systems. An abstract object is the maximal atlas which contains all admissible charts and therefore all conceivable diffeomorphisms between them. An atlas with a small number of charts suffice for practical purposes.

Let us furthermore elaborate on the mathematical definition of vectors. A tangent vector at the point x is defined as a derivative operator at this point with the usual properties. Tangent vectors at the point x span a vector space, the *tangent space* $\mathcal{T}_x\mathcal{M}$. Tangent spaces at different points are isomorphic, but should not be compared directly. The *tangent bundle* is defined as the collection of all tangent spaces

$$\mathcal{T}\mathcal{M} := \{(x, y); x \in \mathcal{M}, y \in \mathcal{T}_x\mathcal{M}\}. \tag{9.17}$$

The tangent bundle is a manifold of twice the dimension of the underlying manifold. It inherits a topological structure from the manifold. This structure is generally not the direct product of the manifold and one of its tangent spaces, i.e. $\mathcal{T}\mathcal{M}$ can have a non-trivial topology. Therefore $\mathcal{T}\mathcal{M}$ is a *fibre bundle* over $\mathcal{M}$ with the projection

$$\pi : \mathcal{T}\mathcal{M} \to \mathcal{M}, \qquad \pi(x, y) = x. \tag{9.18}$$

The role of the tangent bundle is to properly define a vector field F. In principle, we want

$$F(x) \in \mathcal{T}_x\mathcal{M} \tag{9.19}$$

This function has an ill-defined type $F : \mathcal{M} \to \mathcal{T}_?\mathcal{M}$. Instead one defines a vector field as a *section* of the tangent bundle with type

$$F : \mathcal{M} \to \mathcal{T}\mathcal{M}. \tag{9.20}$$

The section must map a point x to the tangent space $\mathcal{T}\mathcal{M}$ at this point. This can be achieved by the constraint

$$\pi \circ F = \mathrm{id}\,. \tag{9.21}$$

This somewhat bloated structure is required for mathematical consistency, but it does not hurt too much to consider $F(x) \in \mathcal{T}_x\mathcal{M}$

9.2 Riemannian Geometry

Riemannian geometry introduces a metric field on a differentiable manifold. This field serves multiple purposes: It introduces a measure of distance and of angles. Furthermore, it relates the tangent spaces at nearby points. This allows to define a measure of curvature.

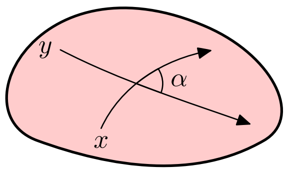
For Riemannian geometry the metric must be positive definite. A theory of spacetime, however, requires a metric of signature $(D-1,1)$. This generalisation of Riemannian geometry is straight-forward and is called pseudo Riemannian geometry. We shall not make a distinction here.

Metric. The metric tensor field $g_{\mu\nu}(x)$ is a tensor of rank $(0,2)$ with symmetric indices μ,ν. Most immediately, it defines the length L for a curve $x^\mu(\tau)$ [4]

$$L = \int d\tau |\dot{x}|, \qquad |\dot{x}| := \sqrt{g_{\mu\nu}(x)\,\dot{x}^\mu\,\dot{x}^\nu}. \tag{9.22}$$

Angles between intersecting curves can be measured as usual via the scalar product

$$\cos\alpha = \frac{g_{\mu\nu}\,\dot{x}^\mu\,\dot{y}^\nu}{|\dot{x}|\,|\dot{y}|}\,. \tag{9.23}$$



More generally, the metric define the length squared ds^2 of infinitesimal line elements dx^μ via

$$ds^2 = g_{\mu\nu}(x)\,dx^\mu\,dx^\nu. \tag{9.24}$$

The metric field has an inverse $g^{\mu\nu}(x)$ with the property

$$g^{\mu\nu}(x) g_{\nu\rho}(x) = \delta^\mu_\rho. \tag{9.25}$$

As in special relativity and Minkowski spacetime, these two instances of the metric are used to lower and raise indices,

$$F_\mu = g_{\mu\nu}F^\nu, \qquad F^\mu = g^{\mu\nu}F_\nu. \tag{9.26}$$

More abstractly, the metric translates between tangent and cotangent spaces. Therefore, there is essentially no distinction between tangent and cotangent vectors, and tensor fields can be classified by their overall rank alone.

[4] The definition of length applies to space-like curves; for time-like curves the notion of length (proper time) requires the opposite different sign under the square root. Light-like curves with $|\dot{x}| = 0$ have no proper length by definition.

Covariant Derivatives. We already pointed out that tangent spaces at different points are unrelated. Therefore, partial derivatives of tensor fields are not tensor fields. This represents be a principal difficulty towards setting up a field theory on curves spaces. It can be overcome by introducing a covariant derivative which acts on a tensor field according to its tensor structure

$$\begin{aligned} D_\mu F &= \partial_\mu F, \\ D_\mu F^\nu &= \partial_\mu F^\nu + \Gamma^\nu_{\mu\rho} F^\rho, \\ D_\mu F_\nu &= \partial_\mu F_\nu - \Gamma^\rho_{\mu\nu} F_\rho, \\ D_\mu F^{\nu\rho}_\sigma &= \partial_\mu F^{\nu\rho}_\sigma + \Gamma^\nu_{\mu\nu'} F^{\nu'\rho}_\sigma + \Gamma^\rho_{\mu\rho'} F^{\nu\rho'}_\sigma - \Gamma^{\sigma'}_{\mu\sigma} F^{\nu\rho}_{\sigma'}, \\ &\ldots . \end{aligned} \tag{9.27}$$

The affine connection $\Gamma^\mu_{\nu\rho}(x)$ has the index structure of a tensor field of rank $(1,2)$, however, its transformation properties are slightly different to absorb the undesired terms in the transformation of derivatives of tensor fields

$$\Gamma'^{\mu'}_{\nu'\rho'}(x') = \frac{\partial x'^{\mu'}}{\partial x^\mu}\,\frac{\partial x^\nu}{\partial x'^{\nu'}}\,\frac{\partial x^\rho}{\partial x'^{\rho'}}\,\Gamma^\mu_{\nu\rho}(x) + \frac{\partial x'^{\mu'}}{\partial x^\sigma}\,\frac{\partial^2 x^\sigma}{\partial x'^{\nu'}\,\partial x'^{\rho'}}\,. \tag{9.28}$$

The above definition of covariant derivative respects contractions of indices in the sense that

$$D_\mu(F^\nu G_\nu) = (D_\mu F^\nu)G_\nu + F^\nu(D_\mu G_\nu) = \partial_\mu(F^\nu G_\nu). \tag{9.29}$$

A further desirable property would be that the lowering and raising of indices commutes with the covariant derivative. This implies that the metric must be covariantly constant

$$D_\mu g_{\nu\rho} = 0. \tag{9.30}$$

This condition imposes strong constraints on the affine connection $\Gamma^\mu_{\nu\rho}$. In particular, the part with symmetric lower indices $\nu\rho$ is fully determined by the metric. To further constrain the antisymmetric part, we can impose a condition on the second derivatives of an arbitrary scalar field F

$$D_\mu D_\nu F = D_\nu D_\mu F. \tag{9.31}$$

This constraint implies the absence of *torsion*. Together the constraints determine the so-called *Christoffel* connection $\Gamma^\mu_{\nu\rho}$ completely

$$\Gamma^\mu_{\nu\rho} = \tfrac{1}{2} g^{\mu\sigma}\big(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\big). \tag{9.32}$$

The transformation properties for this combination as a connection essential follow by construction via covariant equations.

Curvature. The covariant derivative can be used to construct interesting observable a Riemannian manifold, the curvature tensor $R^\mu{}_{\nu\rho\sigma}$. It follows from the commutator of covariant derivatives acting on a vector field F_ρ

$$D_\mu D_\nu F^\rho - D_\nu D_\mu F^\rho = R^\rho{}_{\sigma\mu\nu} F^\sigma. \tag{9.33}$$

More explicitly, it can be expressed in terms of the Christoffel connection

$$R^{\mu}{}_{\nu\sigma\rho} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\kappa}_{\nu\sigma}\Gamma^{\mu}_{\kappa\rho} - \Gamma^{\kappa}_{\nu\rho}\Gamma^{\mu}_{\kappa\sigma}. \tag{9.34}$$

It is called the curvature tensor because the vanishing of all of its components is equivalent to the existence of a local coordinate system where the metric field is constant.

The curvature tensor has a number of useful properties which follow from its definition, compatibility with the metric and the absence of torsion

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\rho\sigma\mu\nu} \tag{9.35}$$

as well as

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0. \tag{9.36}$$

Furthermore the Jacobi identity of covariant derivatives imply the so-called Bianchi identities

$$D_\kappa R_{\mu\nu\rho\sigma} + D_\rho R_{\mu\nu\sigma\kappa} + D_\sigma R_{\mu\nu\kappa\rho} = 0. \tag{9.37}$$

9.3 General Relativity

Let us state the equations of general relativity without further ado.

Einstein Equations. General relativity has a very natural formulation in Riemannian geometry. With a few basic assumptions we can derive a set of field equations for general relativity: The equations should

- ... be manifestly covariant,
- ... involve up to two derivatives of the metric,
- ... couple to matter fields appropriately,
- ... have the desired non-relativistic limit,

The covariant objects (tensor fields) we have at our disposal in pure Riemannian geometry are

- the metric $g_{\mu\nu}$,
- the curvature tensor $R_{\mu\nu\rho\sigma}$,
- covariant derivatives D_μ.

However, the covariant derivatives turn out not to be suitable for our purposes because the metric is covariantly constant and the curvature tensor already involves two derivatives of the metric.

By means of the equivalence principle of inertial and gravitational mass, one can expect that matter fields enter the gravitational equations of motion only through the energy-momentum tensor $T_{\mu\nu}$. This tensor field is covariantly conserved

$$D^\mu T_{\mu\nu} = 0, \tag{9.38}$$

as an expression of conservation of energy and momentum. The energy-momentum tensor is of second order in the derivatives of the matter fields, and therefore should contribute no more than linearly to the equations of motion.

Considering the tensor structures of the various objects, we can make the ansatz[5]

$$R_{\mu\nu} + aRg_{\mu\nu} + bg_{\mu\nu} + cT_{\mu\nu} = 0, \tag{9.39}$$

with three constants a, b, c to be determined. Here $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ is the so-called *Ricci tensor* and $R = g^{\mu\nu} R_{\mu\nu}$ is the *Ricci scalar*. From conservation of the energy-momentum tensor it follows that $a = -\frac{1}{2}$. Furthermore, c must be matched with Newtons constant G as $c = 8\pi G$. Only a fool would set the cosmological constant $b = \Lambda$ to $b = 0$. Altogether we obtain the Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + 8\pi G\, T_{\mu\nu} = 0. \tag{9.40}$$

Einstein–Hilbert action. The above Einstein equations follow from an action functional as the Euler–Lagrange equations. Consider the Einstein–Hilbert action

$$S = \frac{1}{16\pi G} \int d^D x \sqrt{-\det g}\big(R - 2\Lambda\big) + S_{\text{matter}}. \tag{9.41}$$

Note that $\sqrt{-\det g_{..}}$ is not a scalar field although it carries no indices. It is rather a *scalar density field* which transforms as

$$\sqrt{-\det g'_{..}(x')} = \big|\det(\partial x/\partial x')\big| \sqrt{-\det g_{..}(x)}. \tag{9.42}$$

The additional transformation factor compensates the Jacobian factor of the transformation of the measure $d^D x$. Altogether the combination $d^D x \sqrt{-\det g}$ serves as the invariant volume element.

The variation is most conveniently computed using a variational identity of the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} = -\delta g_{\mu\nu} R^{\mu\nu} \tag{9.43}$$

and the variation of a determinant $\delta \det g = \delta g_{\mu\nu} g^{\mu\nu}$. Furthermore, the variation of the matter w.r.t. the metric yields the energy-momentum tensor $\delta S_{\text{matter}} = -\frac{1}{2} \int d^D x \sqrt{-\det g}\, \delta g_{\mu\nu}\, T^{\mu\nu}$. Altogether we obtain

$$-\frac{1}{8\pi G}\big(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu}\big) - T^{\mu\nu} = 0 \tag{9.44}$$

Gravitational Waves. Let us find solutions of the gravitational equations which are small perturbations $h_{\mu\nu}(x)$ of plain Minkowski spacetime

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x). \tag{9.45}$$

Since we are interested in terms linear in the small field h, we can make an ansatz in terms of a single plane wave with momentum p and polarisation tensor $\epsilon_{\mu\nu}$

$$h_{\mu\nu}(x) = \epsilon_{\mu\nu} \exp(ip \cdot x). \tag{9.46}$$

[5]Furthermore, we may or may not demand that the totally traceless part of $R^\mu{}_{\nu\rho\sigma}$ is zero. This does not lead to a desirable field equation for physics.

The resulting linearised Christoffel symbols and curvature tensor and Ricci tensor read

$$\begin{aligned}
\Gamma^\mu_{\nu\rho} &= \tfrac{i}{2}\bigl(p_\nu \epsilon^\mu_\rho + p_\rho \epsilon^\mu_\nu - p^\mu \epsilon_{\nu\rho}\bigr)\exp(ip\cdot x),\\
R^\mu{}_{\nu\sigma\rho} &= \tfrac{1}{2}\bigl(-p_\rho p_\nu \epsilon^\mu_\sigma + p_\rho p^\mu \epsilon_{\nu\sigma} + p_\sigma p_\nu \epsilon^\mu_\rho - p_\sigma p^\mu \epsilon_{\nu\rho}\bigr)\exp(ip\cdot x),\\
R_{\nu\rho} &= \tfrac{1}{2}\bigl(-p_\mu p_\nu \epsilon^\rho_\rho + p_\mu p^\rho \epsilon_{\rho\nu} + p^\rho p_\nu \epsilon_{\mu\rho} - p^2 \epsilon_{\mu\nu}\bigr)\exp(ip\cdot x).
\end{aligned} \tag{9.47}$$

The Einstein equations without matter and cosmological constant boil down to $R_{\mu\nu} = 0$, i.e.

$$-p_\mu p_\nu \epsilon^\rho_\rho + p_\mu p^\rho \epsilon_{\nu\rho} + p^\rho p_\nu \epsilon_{\mu\rho} - p^2 \epsilon_{\mu\nu} = 0. \tag{9.48}$$

For $p^2 \neq 0$ the equation immediately implies that $\epsilon_{\mu\nu}$ must be parallel to either p_μ or p_ν. An ansatz is

$$\epsilon_{\mu\nu} = \xi_\mu p_\nu + \xi_\nu p_\mu. \tag{9.49}$$

This is in fact already a solution which corresponds to the diffeomorphisms $\delta x^\mu = \xi^\mu$ applied to trivial Minkowski space $g_{\mu\nu} = \eta_{\mu\nu}$. Therefore it is a physically trivial solution which we can ignore.

Consequently, we only consider the case $p^2 = 0$ Let us furthermore define $\pi_\mu = p^\rho \epsilon_{\mu\rho}$. Our equation then reads

$$-p_\mu p_\nu \epsilon^\rho_\rho + p_\mu \pi_\nu + p_\nu \pi_\mu = 0. \tag{9.50}$$

This implies that π must be parallel to p. By means of a diffeomorphism we can actually remove the the component of π parallel to p, and therefore set $\pi = 0$ without restrictions on generality. The remaining equation demands that ϵ is traceless. Consequently, the physical plane wave solutions of general relativity are enumerated by traceless, symmetric tensors $\epsilon_{\mu\nu}$ modulo p_μ which are orthogonal to p^μ. Altogether these are $D(D-3)/2$ degrees of freedom which form a symmetric traceless representation of $\mathrm{SO}(D-2)$.

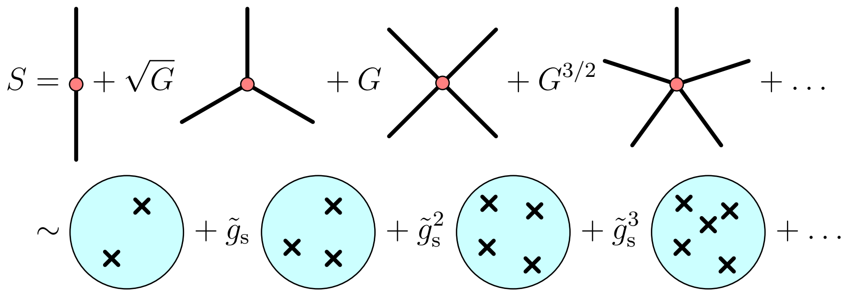
Graviton Scattering. The quantisation of gravitational waves leads to graviton excitations. One can expand around flat background with gravitons using the above weak coupling ansatz

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}(x). \tag{9.51}$$

Here we have inserted the Newton constant in order to achieve a canonically normalised kinetic term. Since the Einstein–Hilbert action is non-polynomial in $g_{\mu\nu}$ due to the occurrence of the inverse field $g^{\mu\nu}$ and due to the term $\sqrt{-\det g}$, one obtains a tower of interaction terms for the graviton field $h_{\mu\nu}(x)$

$$S \simeq S_2 + G^{1/2}\, S^3 + G\, S^4 + G^{3/2}\, S^5 + \ldots . \tag{9.52}$$

Scattering of four gravitons therefore has an effective coupling constant of G. This is to be compared to the corresponding process in string theory where one finds $\tilde{g}_\mathrm{s}^2 := g_\mathrm{s}^2 \kappa^{D-2}$. The factor of κ^{D-2} is required to match the mass dimension mass dimension $2-D$ of Newton's constant, hence κ defines the Planck scale up to factors of g_{s}.



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10 String Backgrounds

We start with contemplations on the connection of string theory and general relativity:

- We have seen that the string spectrum contains the graviton. This graviton interacts according to the laws of general relativity (up to stringy corrections at higher orders) which is a theory of spacetime geometry.
- So far we have assumed that strings move in a flat background with canonical coordinates. However, strings can also move in a curved background described by general relativity.

How do these two connections fit together?

- Should we quantise the string background?
- Is the string graviton the same as the Einstein graviton?
- Is there a back-reaction between strings and gravity?

10.1 Graviton Vertex Operator

Let us compare the graviton as a string excitation and as an excitation of the background metric. We assume it has momentum q and polarisation tensor $\epsilon_{\mu\nu}$.

Vertex Operator Construction. The graviton is represented by the closed string state

$$|\epsilon; q\rangle = \epsilon_{\mu\nu}\big(\alpha^{\mathrm{L},\mu}_{-1}\alpha^{\mathrm{R},\nu}_{-1} + \alpha^{\mathrm{L},\nu}_{-1}\alpha^{\mathrm{R},\mu}_{-1}\big)|0; q\rangle. \tag{10.1}$$

The corresponding vertex operator in the CFT description reads

$$\begin{aligned}\mathcal{O}^{\mu\nu} &= {:}(\partial X^\mu \bar\partial X^\nu + \partial X^\nu \bar\partial X^\mu)e^{iq\cdot X}{:}\\ &\sim {:}\sqrt{\det -g}\, g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\, e^{iq\cdot X}{:}.\end{aligned} \tag{10.2}$$

In the second line we have reinstated the worldsheet metric to make the expression valid outside the conformal gauge. Insertion of the vertex operator into the string worldsheet yields the expression

$$V = \int d^2\xi\, \tfrac{1}{2}\epsilon_{\mu\nu}\mathcal{O}^{\mu\nu}. \tag{10.3}$$

Background Metric Construction. We start with a flat background metric and perturb it by a plane wave

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + \epsilon_{\mu\nu}e^{iq\cdot x} + \ldots. \tag{10.4}$$

The coupling of strings to generic background is straight-forward by the replacement $\eta_{\mu\nu} \to G_{\mu\nu}$ in the worldsheet action

$$S = -\frac{1}{2\pi\kappa^2} \int d^2\xi \sqrt{-\det g}\, g^{\alpha\beta}\, \tfrac{1}{2} G_{\mu\nu}(X)\, \partial_\alpha X^\mu\, \partial_\beta X^\nu. \qquad (10.5)$$

We notice that a string in a background with a weak gravitational wave has the same action as a string in flat space deformed by the graviton vertex operator

$$S = S_0 - \frac{1}{2\pi\kappa^2}\, V + \ldots . \qquad (10.6)$$

Conclusion. We conclude that the graviton mode of the string spectrum is equivalent to a gravitational wave in the background on which the string propagates. There is no conflict between the two ways in which gravity appear in string theory; they are actually the same.

We can thus argue as follows: The quantum string on flat space contains gravitons as excitations. The gravitons introduce curvature and thus deform flat background. Therefore string theory contains quantum gravity upon quantisation. This is the picture for a small number of excitations and small deformations of flat space. Large deformations away from a flat background should be represented by coherent states of a large number of gravitons.[1]

How about the dependence on the background on which string theory is formulated? Due to the appearance of graviton modes, string quantisation automatically probes nearby backgrounds. The low-energy stringy physics will thus see the classical background and nearby geometries. However, the full quantum string theory can be expected to be independent of the background because it contains all backgrounds as different states.[2]

10.2 Curved Backgrounds

Consider strings propagating on a curved background described by the metric field $G_{\mu\nu}(x)$; a curious insight awaits.

The action in conformal gauge reads

$$S = -\frac{1}{2\pi\kappa^2} \int d^2\xi\, \tfrac{1}{2} G_{\mu\nu}(X)\, \eta^{\alpha\beta} \partial_\alpha X^\mu\, \partial_\beta X^\nu. \qquad (10.7)$$

For a generic metric G, the equations of motion for X are non-linear. This type of model is called a *non-linear sigma model.* The string background is called the

[1] This is a leap of faith, but it should work somewhat analogously to the electromagnetic field. For instance, the particle picture of photons describes the photo-electric effect very well, whereas the generation of a Coulomb potential around a charged object is hardly intuitive in this picture.

[2] This is essentially the same issue as background dependence in quantum gravity. Note, however, that the asymptotic behaviour of the background may have a relevant influence on the quantum string and quantum gravity theory because it would take an infinite amount of energy to change the geometry asymptotically.

target space of the model. The metric field $G_{\mu\nu}(x)$ acts as the coupling constants for the model. In fact, there are infinitely many coupling constants contained in the field $G(x)$ when Taylor expanding it around some point x.

In most QFT's, coupling constants are renormalised upon quantisation. The problem here is:

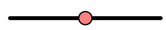
- The classical action has conformal symmetry.
- Conformal symmetry is indispensable to remove some degrees of freedom.
- The renormalised coupling $G(x, \mu)$ depends on a scale μ which is introduced by the quantisation or regularisation process.
- This new scale breaks quantum conformal invariance of the string. There may be a conformal anomaly threatening consistency of the quantum string!

Background Field Method. Let us compute the conformal anomaly. A well-suited method is background field quantisation:

- Pick a (reasonably simple) classical solution X_0 of the string equations of motion.
- Then add fluctuations to this field $X = X_0 + \kappa Y$. The field Y is considered to be the *quantum field* while X_0 remains a classical field.[3]

Next, expand the action $S[X] = S[X_0] + Y^2 + \kappa Y^3 + \ldots$ in orders of Y (or κ).

- The value of the classical action $S[X_0]$ at order Y^0 does not matter for the quantum theory.
- There is no linear term in Y due to the classical equations of motion for X_0.
- The second order Y^2 serves as the kinetic term for the quantum field Y.



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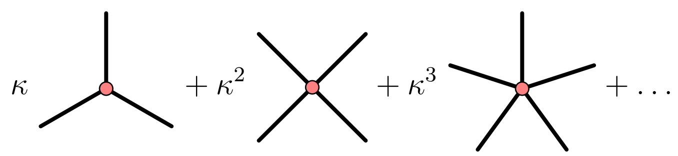
- The higher orders $Y^3, Y^4, \ldots$ are cubic, quartic, … interactions of the quantum fields.



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We can now use a trick to simplify the interactions slightly for our purposes: We use target space diffeomorphisms such that locally, at a particular point in spacetime x, the metric is stationary.[4] This is convenient because the quadratic terms are governed by the target space curvature tensor

$$G_{\mu\nu}(x + \delta x) = G_{\mu\nu}(x) + \tfrac{1}{3} R_{\mu\rho\nu\sigma}(x)\, \delta x^\rho\, \delta x^\sigma + \ldots . \tag{10.10}$$

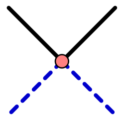
[3] It can also be viewed as the expectation value of the field X.

[4] Consideration of a single point is sufficient for the leading order divergence. For higher orders we should make sure that the metric is stationary everywhere one the classical worldsheet solution. This can be achieved for any geodesic curve (point-like string) and potentially for fully geodesic string configurations. Otherwise one will have to deal with a few additional interaction terms.

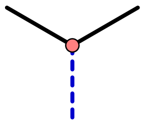
Focusing on this point, we will find fewer interactions and our results are automatically covariant. To quadratic order in Y we then find only two terms

$$S_2 = -\int \frac{d^2\xi}{2\pi}\,\eta^{\alpha\beta}\left(G_{\mu\nu}\partial_\alpha Y^\mu \partial_\beta Y^\nu + \tfrac{1}{3}R_{\mu\rho\nu\sigma}\partial_\alpha X_0^\mu \partial_\beta X_0^\nu Y^\rho Y^\sigma\right). \tag{10.11}$$

Here we can consider the first term in the quadratic action S_2 as the kinetic term for the quantum field Y and the second term as a perturbation. To this end, we use the field ∂X_0 as another formal expansion parameter, and there is only one interaction vertex at order κ^0. In the corresponding graph we shall mark the classical field ∂X by a dashed line.


(10.12)

Note that without the above trick, there is an additional vertex to be taken into account, which leads to many more graphs to be considered.[5]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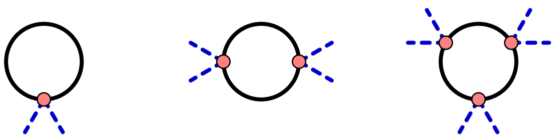
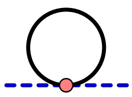
One-Loop Renormalisation. Let us now compute one-loop quantum corrections to the classical action $S[X_0]$ in order to understand the regularisation issues. When integrating out the quantum field Y at κ^0, we obtain a tower of bubbles with an increasing number of insertions of ∂X.


$\ldots$ (10.14)

Among these contributions, the only UV divergent term by power counting is the first one. Let us therefore investigate it more closely. Fixing the location ξ of the vertex on the worldsheet,[6] the diagram evaluates to

$$R_{\mu\rho\nu\sigma}\partial_\alpha X_0^\mu(\xi)\partial^\alpha X_0^\nu(\xi)\big\langle Y^\rho(\xi)Y^\sigma(\xi)\big\rangle. \tag{10.15}$$



Here, the two-point correlator of quantum fields depends strongly on the background metric $G_{\mu\nu}$. However, in the ultraviolet region $\xi_1 \to \xi_2$ the dependence on the metric trivialises and we can trust the results of CFT

$$\big\langle Y^\rho(\xi_1)Y^\sigma(\xi_2)\big\rangle \simeq -G^{\rho\sigma}\log|\xi_1-\xi_2|. \tag{10.16}$$

[5]It may be necessary to include this vertex at higher orders because the above simplification works only at one specialised point.

[6]At this point the field $X^\mu(\xi)$ should point to the location where the metric $G_{\mu\nu}(X)$ simplifies.

So we see that the diagram has a logarithmic divergence whose structure $\partial_\alpha X_0^\mu \partial^\alpha X_0^\nu$ matches the structure of the classical action $S[X_0]$. Therefore we can regularise the theory by renormalising $G_{\mu\nu}$. This makes $G_{\mu\nu}(x)$ a running coupling constant with one-loop beta function

$$\frac{\mu\partial}{\partial\mu} G_{\mu\nu} = \beta_{\mu\nu} = \kappa^2 R_{\mu\nu}, \qquad R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}. \tag{10.17}$$

Anomaly. Dependence on a scale evidently breaks conformal symmetry: We find that the stress-energy tensor has acquired a trace after renormalisation which is proportional to the above beta function

$$\eta^{\alpha\beta} T_{\alpha\beta} = -\frac{1}{2\kappa^2} \beta_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu. \tag{10.18}$$

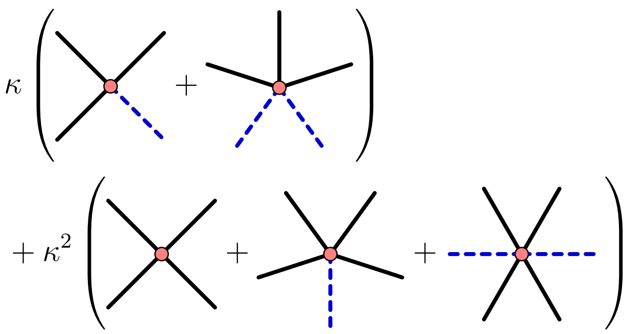
This describes the anomaly of Weyl symmetry!

We know that Weyl symmetry is essential for obtaining the correct degrees of freedom for string theory. The only way to remove them and make string theory consistent is to set $\beta_{\mu\nu} = 0$. Surprisingly, this yields the Einstein equation

$$R_{\mu\nu} = 0. \tag{10.19}$$

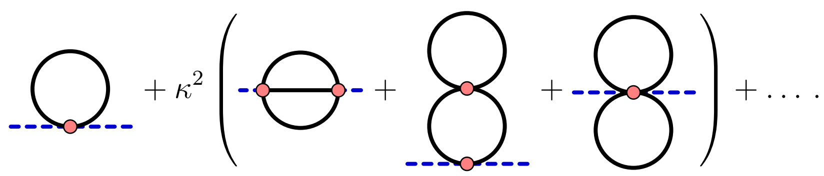
In other words, quantum strings can propagate only on Einstein backgrounds. We have thus found another way to recover general relativity from string theory, and we gain confidence that the spin-2 particles at level one are in fact gravitons.

Higher Corrections. To understand higher-order corrections in κ, let us list the relevant vertices.



(10.20)

The divergent corrections to the classical action at higher perturbative orders in κ^2 take the form[7]



(10.21)

[7] This set of diagrams is based on the assumption of a vanishing Christoffel symbol on the worldsheet. Several further diagrams may contribute otherwise.

They amount to the following two-loop beta function[8]

$$\beta_{\mu\nu} = \kappa^2 R_{\mu\nu} + \tfrac{1}{2}\kappa^4 R_{\mu\rho\sigma\kappa} R_\nu{}^{\rho\sigma\kappa} + \ldots\,. \tag{10.22}$$

Absence of the conformal anomaly, $\beta_{\mu\nu} = 0$, implies that in string theory the Einstein equations receive corrections at the Planck scale.

10.3 Form Field and Dilaton

What about the other (massless) fields: two-form $B_{\mu\nu}$ and dilaton scalar Φ? If we view them as fields in/of the background manifold, can we couple them to the string action as well?

The two-form naturally couples via antisymmetric combination of worldsheet derivatives

$$\frac{1}{2\pi\kappa^2}\int d^2\xi\, \tfrac{1}{2}B_{\mu\nu}(X)\,\varepsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu = \frac{1}{2\pi\kappa^2}\int B. \tag{10.23}$$

In fact, this is the canonical coupling of a two-dimensional worldsheet to a two-form in direct analogy to the coupling of a charged particle worldline to the electromagnetic field. The string carries a two-form charge as we shall see a little later.

The dilaton couples to the worldsheet Riemann scalar

$$\frac{1}{4\pi}\int d^2\xi\, \sqrt{-\det g}\, \Phi(X)\, R[g]. \tag{10.24}$$

This is interesting for several reasons:

- The Euler characteristic χ of the worldsheet appears.
- The coupling is not Weyl invariant.
- The scalar can mix with gravity.
- We can get away from 26 dimensions.

Let us discuss them in more detail below.

Low-Energy Effective Action. First we discuss the various beta functions (trace of the renormalised stress-energy tensor T) which appear in string theory coupled to a generalised background consisting of G, B and Φ

$$\begin{aligned} g^{\alpha\beta}T_{\alpha\beta} = &-\frac{1}{2\kappa^2}\big(\sqrt{-\det g}\,\beta^G_{\mu\nu}g^{\alpha\beta} + \beta^B_{\mu\nu}\varepsilon^{\alpha\beta}\big)\partial_\alpha X^\mu \partial_\beta X^\nu \\ &- \tfrac{1}{2}\beta^\Phi R[g], \end{aligned} \tag{10.25}$$

where

$$\begin{aligned} \beta^G_{\mu\nu} &= \kappa^2 R_{\mu\nu} + 2\kappa^2 D_\mu D_\nu \Phi - \tfrac{1}{4}\kappa^2 H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma}, \\ \beta^B_{\mu\nu} &= -\tfrac{1}{2}\kappa^2 D^\lambda H_{\mu\nu\lambda} + \kappa^2 D^\lambda \Phi H_{\mu\nu\lambda}, \\ \beta^\Phi &= -\tfrac{1}{2}\kappa^2 D_\mu D^\mu \Phi + \kappa^2 D^\mu \Phi D_\mu \Phi - \tfrac{1}{24}\kappa^2 H_{\mu\nu\rho}H^{\mu\nu\rho}. \end{aligned} \tag{10.26}$$

[8]Note that there are also corrections from the expansion in the string coupling g_s which we do not consider here.

Quantum consistency of (conformal symmetry in) string theory requires $\beta^G = \beta^B = \beta^\Phi = 0$. These are the standard equations for a graviton, a two-form field and a scalar. They follow from an action in spacetime

$$S \sim \int d^{26}x \sqrt{-\det G}\, e^{-2\Phi} \left(R - \tfrac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial^\mu \Phi\, \partial_\mu \Phi\right). \tag{10.27}$$

This is the string theory low-energy effective action. It encodes the low-energy physics of string theory viewed from the perspective of the background. Note that there are further corrections from curvature (κ) and loops (g_s) not displayed here.

The anomaly equations have the trivial solution $G = \eta$, $B = 0$, $\Phi = \Phi_0$ which describes a flat background. The solution equally applies to torus compactification to effectively reduce the number of dimensions.

String Coupling. Suppose $\Phi = \Phi_0$ is constant, then the dilaton coupling term is topological

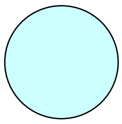
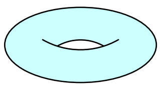
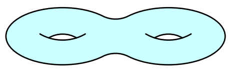
$$\int d^2\xi \sqrt{-\det g}\, R[g] \sim \chi. \tag{10.28}$$

It measures the Euler characteristic $\chi = 2h - 2$ of worldsheet.

Set $g_\mathrm{s} = e^{i\Phi_0}$. Then the generating functional yields precisely χ factors of g_s.

$$e^{iS} \simeq e^{i\Phi_0 \chi} = g_\mathrm{s}^\chi. \tag{10.29}$$

The expansion in this g_s thus corresponds to an expansion in worldsheet topology

g_s^{-2}  $+\, g_\mathrm{s}^0$  $+\, g_\mathrm{s}^2$  $+ \ldots .$ (10.30)

For general curved backgrounds, the string coupling g_s is determined by the asymptotic behaviour of the background, namely the asymptotic value Φ_0 of dilaton field Φ.

String Frame. Notice the unusual factor of $\exp(-2\Phi)$ in the above effective action S.

The scalar degrees of freedom can mix with the gravitational degrees of freedom. We may as well define a rescaled metric

$$G'_{\mu\nu} = f(\Phi) G_{\mu\nu}. \tag{10.31}$$

With a suitable choice of f we can remove the factor $\exp(-2\Phi)$. We can go from the so-called *string frame* with $\exp(-2\Phi)$ to the so-called *Einstein frame* where we recover the canonical kinetic term for each field.

Non-Critical Strings. We have seen earlier that $D \neq 26$ breaks Weyl symmetry of the string action. Furthermore it is broken by the above anomalies. Therefore both anomalies should appear in the same place. In fact, D enters in the effective action as the worldsheet cosmological constant

$$S = \ldots \bigl(R - \tfrac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial^\mu \Phi\, \partial_\mu \Phi - \tfrac{2}{3}\kappa^{-2}(D-26)\bigr)\,. \qquad (10.32)$$

This implies that we can have $D < 26$, but it requires a spacetime curvature at the Planck scale.

Dilaton Scaling. We note two further peculiarities: The dilaton coupling to the worldsheet is not Weyl invariant and it has an unconventional power of κ.

The factor of κ moves the classical Weyl breakdown effectively to one loop. There it can cancel against the quantum anomalies of other fields. Together this furnishes a consistent choice of normalisation.

10.4 Open Strings

The introduction of open strings leads to additional states, fields and couplings which complete the geometric picture of D-branes in string theory in an exciting way.

- There are additional string states; e.g. the massless vectors (photon):

$$|\zeta; q\rangle = \zeta_\mu \alpha^\mu_{-1} |0; q\rangle. \qquad (10.33)$$

- They correspond to additional vertex operators defined on the ends of the string; e.g. for the photon

$$V[\zeta, q] \sim \int d\tau\, \zeta_\mu \partial_\tau X^\mu \, \exp(iq \cdot X). \qquad (10.34)$$

- The additional fields of the background geometry couple to the ends of the string. They can be identified as for closed strings such that adding the vertex operator to the action has the same effect as switching on a background field.

Evidently, the coupling depends on the choice of string boundary conditions. We have already seen that the latter can be identified with Dp-branes. Let us therefore discuss the arising Neumann and Dirichlet boundary conditions.

Neumann Boundaries. Consider the coordinates X^a, $a = 0, \ldots, p$, with Neumann conditions: We couple an abelian one-form gauge field A to the end of a string

$$\int_{\text{end}} d\tau \dot X^a A_a(X) = \int_{\text{end}} A. \qquad (10.35)$$

- This is the natural coupling of a charged point-particle to a gauge field A.
- The string end is a charged point-like object with a worldline.

Note: The gauge field A_a needs to exist only on the Dp-brane to which the string ends are constrained.

The classical coupling of A respects Weyl symmetry. The quantum anomaly is described by a beta function

$$\beta_a^A \sim \kappa^4 \partial^b F_{ab}. \tag{10.36}$$

Absence of the conformal anomaly requires the Maxwell equation $\partial^b F_{ab} = 0$ to hold. Consequently, the associated low-energy effective action reads

$$S \sim -\kappa^4 \int d^{p+1}x\, \tfrac{1}{4} F_{ab} F^{ab}. \tag{10.37}$$

It is an action in the $(d+1)$-dimensional worldvolume of the Dp-brane.

For planar Dp-branes in a flat background we can also include higher corrections in κ. This leads to the exact Born–Infeld action

$$S \sim \int d^{p+1}x \sqrt{-\det(\eta_{ab} + 2\pi\kappa^2 F_{ab})}\,. \tag{10.38}$$

The leading order is the Maxwell kinetic term, but there are corrections at higher orders in κ.

Dirichlet Boundaries. The coupling of the Dirichlet directions X^m, $m = p+1, \ldots, D-1$, is rather different:

- The field X^m is fixed, but we can use X'^m for the coupling to background fields.
- The corresponding background field Y_m describes displacement of the Dp-brane in transverse directions.
- Dp-branes are in fact dynamical objects!

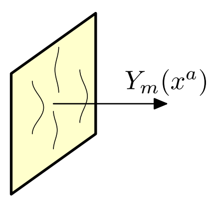
The beta function at leading order describes a collection of massless scalars

$$\beta_m^Y \sim \partial^a \partial_a Y_m. \tag{10.39}$$

The leading order action for these degrees of freedom is evident. The effective action for planar Dp-branes at higher orders is the Dirac action

$$S \sim \int d^{p+1}x \sqrt{-\det(g_{ab})}\,, \tag{10.40}$$

which measures the volume of the Dp-brane via the induced worldvolume metric $g_{ab} = \partial_a Y^m\, \partial_b Y_m$. This form of action clearly identifies the field Y_m as a displacement of the Dp-brane away from its classical planar configuration.



(10.41)

D-Branes Effective Actions. In fact, one can combine the actions for all open string effective degrees of freedom into the so-called Dirac–Born–Infeld action

$$S \sim \int d^{p+1}x \sqrt{-\det(g_{ab} + 2\pi\kappa^2 F_{ab})}\,. \tag{10.42}$$

This is a combination of

- the Dirac action for the dynamics of p-branes and
- the Born–Infeld action for gauge fields on the p-brane.

We can even add the effect of all the closed string fields to the p-brane action

$$S \sim \int d^{p+1}x\, e^{-\Phi} \sqrt{-\det(g_{ab} + 2\pi\kappa^2 F_{ab} + B_{ab})}\,. \tag{10.43}$$

- g_{ab} is the induced metric from the curved background.
- B_{ab} is the pull back of the 2-form field $B_{\mu\nu}$ to the Dp-brane.
- The combination $2\pi\kappa^2 F_{ab} + B_{ab}$ is gauge invariant.
- The dilaton couples as a prefactor as for the closed string.

Coincident Branes. For a single D-brane, the vector field A_a has an associated U(1) symmetry. For N coincident D-branes the gauge group enlarges from $\mathrm{U}(1)^N$ to $\mathrm{U}(N)$.

The non-abelian gauge field should couple to the end of a string via a Wilson line

$$\mathrm{T}\exp \int_{\mathrm{end}} A. \tag{10.44}$$

The resulting effective action at leading order is

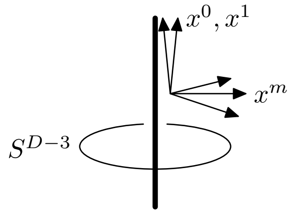
$$S \sim \int d^{p+1}x\, \mathrm{tr}\left(-\tfrac{1}{4}(F_{ab})^2 + \tfrac{1}{2}(D_a Y_m)^2 + \tfrac{1}{4}[Y_m, Y_n]^2\right). \tag{10.45}$$

This is a Yang–Mills action coupled to massless adjoint scalars with quartic interactions.

10.5 Two-Form Field of a String

We have seen that strings couple to various fields. These couplings can be interpreted as interactions between charged objects and gauge fields. As an analogy we use a charged point particle: On the one hand, it generates a Coulomb potential around itself. On the other hand, the electromagnetic field influences its motion. These two phenomena, reaction and back-reaction, are required for consistent interactions. In this picture, the string carries a two-form charge which couples to the two-form gauge field, but which also generates a field configuration around it. In the following we will present this field configuration.

Fundamental String. Consider an infinite straight string stretched along the x^0 and x^1 directions with $x^m = 0$, $m = 2, \ldots, D-1$.



(10.46)

As a charged object, the string configuration generates a two-form potential of the form

$$B = \big(f(x^m)^{-1} - 1\big) dx^0 \wedge dx^1, \tag{10.47}$$

where $f(x^m)$ is a function of the displacement x^m to the string. The interactions with the metric G and the dilaton Φ require

$$ds^2 = f(x^m)^{-1} ds_2^2 + ds_{D-2}^2, \qquad e^{2\Phi} = f(x^m)^{-1}. \tag{10.48}$$

The function f with $r^2 = x_2^2 + \ldots + x_{D-1}^2$ reads[9]

$$f = 1 + \frac{g_s^2 N \kappa^{D-4}}{r^{D-4}}\,. \tag{10.49}$$

One can convince oneself that these background fields B, G, Φ satisfy the low-energy effective string equations of motion essentially because f is a harmonic function.

However, there is an important subtlety: The background field equations are satisfied everywhere except at $r = 0$ where there is a residual term of the form $\delta^{D-2}(x^m)$. The latter represents a source term for the charge of the string.

- The equations of motion follow from the combination of the spacetime action and the worldsheet coupling to the two-form

$$\int_D H \wedge *H + \int_2 B. \tag{10.50}$$

  The source term $\delta^{D-2}(x^m)$ is absorbed by the contribution of the string worldsheet.
- The charge of a string can be measured by the Gauss law via the field $*H$. We put a $(D-3)$-dimensional sphere at fixed r to enclose a time-slice of the string

$$Q = \int_{D-3} *H = N. \tag{10.51}$$

  The above string has N units of charge. One can show that this number must be quantised in integers in analogy to the Dirac charge quantisation condition. It is interpreted as the number of strings residing at the plane $x^m = 0$.

Note: A string is the same as a 1-brane, but the above 1-brane is not a D-brane: It originates from closed strings alone; it has nothing to do with open strings, in particular, open strings cannot end on it. The 1-brane is the string itself, it is therefore called the fundamental string or the fundamental 1-brane.

[9] Solutions $f(x^m)$ with more than one centre are also permissible.

Magnetic Brane. An analogous solution of the string effective equations of motion describes a $(D-5)$-brane. It uses a dual $(D-4)$-form potential C defined through

$$H = dB, \qquad *H = dC. \tag{10.52}$$

It carries a magnetic charge

$$Q = \int_3 H. \tag{10.53}$$

The source is located on the $(D-5)$-brane(s). The coupling of $(D-5)$-branes to C compensates the residual source terms. This object is dual to to the fundamental string and therefore called the magnetic M$(D-5)$-brane.

11 Superstrings

Until now, we have encountered only bosonic d.o.f. in string theory. Conversely, matter in nature is dominantly fermionic. We therefore need to add fermions to string theory.

This has several interesting consequences:

- Supersymmetry is inevitable.
- Critical dimension is reduced from $D = 26$ to $D = 10$.
- Stability is increased.
- The closed string tachyon is absent. We also find stable D-branes.
- There are several formulations related by dualities.

11.1 Supersymmetry

We have seen that string theory always includes spin-2 particles which have to be gravitons. A fermionic extension of string theory will likely include spin-$\frac{3}{2}$ particles. These must be gravitini which can only exist in a *supergravity* theory, a supersymmetric version of gravity. Here the spacetime symmetries are extended by supersymmetry.

Super-Poincaré Algebra. The super-Poincaré algebra is an extension of the Poincaré algebra. The Poincaré consists of Lorentz rotations $M_{\mu\nu}$ and translations P_μ,

$$[M, M] \sim M, \qquad [M, P] \sim P, \qquad [P, P] = 0. \tag{11.1}$$

The super-Poincaré algebra has additional generators, the so-called *supercharges* Q^I_m. They transform in spinor representations of the Lorentz algebra (m is a spinor index), and they are *odd* generators whose Lie brackets are *symmetric* rather than anti-symmetric

$$[M, Q] \sim Q, \qquad [Q, P] = 0, \qquad \{Q^I_m, Q^J_n\} \sim \delta^{IJ} \gamma^\mu_{mn} P_\mu. \tag{11.2}$$

Furthermore, $\mathcal{N}$ is the rank of supersymmetry $I = 1, \ldots, \mathcal{N}$.

The supercharges Q relate particles of

- different spin,
- different statistics,
- otherwise equal features.

Therefore, supersymmetry can be viewed as a symmetry that relates "forces" and "matter".

Wess–Zumino Model. Let us discuss a simple example of a supersymmetric field theory in $D=4$ dimensions. It consists of a complex scalar field ϕ and a chiral fermion ψ_m with Lagrangian

$$\begin{aligned}\mathcal{L} = &- \partial^\mu\bar\phi\,\partial_\mu\phi - \bar m m\bar\phi\phi\\ &- i\sigma_\mu^{n\dot m}\bar\psi_{\dot m}\partial^\mu\psi_n - \tfrac12 m\varepsilon^{mn}\psi_m\psi_n + \tfrac12\bar m\varepsilon^{\dot m\dot n}\bar\psi_{\dot m}\bar\psi_{\dot n}\\ &- \tfrac12\bar g m\bar\phi^2\phi - \tfrac12 g\bar m\bar\phi\phi^2 - \tfrac14 g\bar g\bar\phi^2\phi^2\\ &- \tfrac12 g\varepsilon^{mn}\phi\psi_m\psi_n + \tfrac12\bar g\varepsilon^{\dot m\dot n}\bar\phi\bar\psi_{\dot m}\bar\psi_{\dot n}. \end{aligned} \tag{11.3}$$

The fields both have mass $|m|$ and their scalar and Yukawa interactions are governed by a complex coupling constant g. The 2×2 matrices σ_μ are a generalisation of the Pauli matrices to chiral spinors in $D=(3,1)$. The 2×2 matrices ε are anti-symmetric with $\varepsilon^{12}=\varepsilon_{12}=+1$.

Evidently, this model is Poincaré-invariant. In addition it has an invariance parametrised by a constant fermionic chiral spinor $\delta\epsilon_m$

$$\begin{aligned}\delta\phi &= \varepsilon^{mn}\delta\epsilon_m\psi_n,\\ \delta\bar\phi &= -\varepsilon^{\dot m\dot n}\delta\bar\epsilon_{\dot m}\bar\psi_{\dot n},\\ \delta\psi_m &= -i\varepsilon_{mn}\sigma_\mu^{n\dot p}\delta\bar\epsilon_{\dot p}\partial^\mu\phi - \delta\epsilon_m(\bar m\bar\phi + \tfrac12\bar g\bar\phi^2),\\ \delta\bar\psi_{\dot m} &= +i\varepsilon_{\dot m\dot n}\sigma_\mu^{p\dot n}\delta\epsilon_p\partial^\mu\bar\phi - \delta\bar\epsilon_{\dot m}(m\phi + \tfrac12 g\phi^2). \end{aligned} \tag{11.4}$$

To show invariance under this transformation is tedious but straight-forward.

Supermultiplets. The classification of unitarity irreducible representations (UIR) of the Poincaré algebra distinguishes particles by their mass and spin.[1] UIR's of the super-Poincaré algebra combine several UIR's of the ordinary Poincaré algebra into a *supermultiplet*. Let us discuss some features of supermultiplets.

The size and constituents of a supermultiplet depend strongly on the number of dimensions D and the rank of supersymmetry $\mathcal{N}$. This makes a general treatment difficult, but there are some general features.

It is convenient to start with a constituent of minimum spin component under some rotation, the bottom state of the supermultiplet. Half of the supercharges have a corresponding negative spin component, and therefore must annihilate the bottom state. The other half of the supercharges generate the supermultiplet. The supermultiplet is finite because the supercharges anti-commute (up to the momentum generator). For N supercharges in total, a minimal supermultiplet has $2^{N/2}$ constituent fields.

For instance, the Wess–Zumino model has 4 on-shell degrees of freedom: a complex scalar ϕ and a chiral spinor ψ. It has the minimum amount of $\mathcal{N}=1$ supersymmetry which in $D=4$ amounts to $N=4\mathcal{N}=4$ supercharges. The fields ϕ and ψ therefore form the minimal supermultiplet.

Note that in the massless case supermultiplets typically much smaller. It can be shown that half of the supercharges generate states of zero norm. As usual, these

[1] The notion of spin for massless particles is slightly different.

states are irrelevant for physics and should be projected out. The resulting minimal supermultiplet then has only $2^{N/4}$ states.

In the massless Wess–Zumino model there are as many states as in the massive case. However, the supersymmetry transformations for $m = 0$ do not transform between (ϕ, ψ) and $(\bar{\phi}, \bar{\psi})$. Hence the massive supermultiplet of $4 = 2^2$ fields splits up into two massless supermultiplets of $2 = 2^1$ fields each.

The above considerations also lead to an upper bound on the allowable number of supercharges in an interacting QFT: Every positive supercharge increases the considered spin components by half a unit. Given N supercharges, the spin component therefore increases by $N/8$ units in total within the supermultiplet. Therefore the maximum total spin in a supermultiplet is at least $N/16$. Since the maximum spin must not exceed 1/2 for matter fields, 1 for gauge fields and 2 for gravity, the maximum number of allowable supercharges equals 8, 16 and 32, respectively.[2]

Superspace. Supersymmetry is not merely a curiosity of particular QFT's, it also has a geometric meaning: It is the translational symmetry of *superspace* which extends the ordinary Minkowski space coordinates x^μ by N anti-commuting coordinates θ_I^m. Fields on superspace can be expanded in θ to yield a finite collection of fields with various spin

$$F(x, \theta) = F_0(x) + F_m^I(x)\theta_I^m + *\theta^2 + \ldots + F_N(x)\theta^N. \tag{11.5}$$

This allows to package a supermultiplet of particles into a single field on superspace.

Spinors. Supersymmetry is based on spinor representations of the group $\mathrm{Spin}(D-1, 1)$ which is a two-fold cover of the Lorentz group $\mathrm{SO}(D-1, 1)$. Let us discuss spinor representations in various dimensions.

In $D = 4$, the default (Dirac) spinor has 4 complex components, it belongs to the space $\mathbb{C}^4$. It can be split into two chiral spinors (Weyl): $\mathbb{C}^2 \oplus \mathbb{C}^2$. Alternatively, a reality condition (Majorana) can be imposed: $\mathrm{Re}(\mathbb{C}^4) = \mathbb{R}^4 \simeq \mathbb{C}^2 = \mathrm{Re}(\mathbb{C}^2 \oplus \bar{\mathbb{C}}^2)$. Note that in $D = 4$ the reality condition cannot be imposed on chiral spinors because complex conjugation flips chirality.[3]

Let us now summarise the features on spinors in higher dimensions:

- The dimension of the Dirac spinor is multiplied by a factor of 2 for every step $D \to D + 2$.
- Chiral spinors (Weyl) exist only when D is even.
- Real spinors (Majorana) exist for $D = 0, 1, 2, 3, 4 \pmod 8$.
- Real chiral spinors (Majorana–Weyl) exist only for $D = 2 \pmod 8$.

[2]For massive supermultiplets, the number of allowable supercharges naively is $N = 4$, but in the presence of central charges, this number may increase to $N = 8$.

[3]Hence a real spinor is equivalent to a chiral spinor.

The number of supercharges N equals the dimension of a spinor multiplied by the rank $\mathcal{N}$ of supersymmetry.[4] This leads to a maximum rank of supersymmetry $\mathcal{N}$ in a given dimension D and a maximum dimension D for interacting supersymmetric QFT's:

- For $D = 4$ the maximum rank is $\mathcal{N} = 2, 4, 8$ for matter, gauge and gravity theories, respectively.
- In $D = 10$ the minimum spinor is the real chiral spinor with 16 components. This is the dimensional bound for gauge theories.
- In $D = 11$ the real spinor has 32 components. This is the dimensional bound for gravitational theories.

Super Yang–Mills Theory. Let us give a sketch of pure $\mathcal{N} = 1$ supersymmetric gauge theory in $D = 10$ Minkowski space.[5] This theory appears as the low-energy limit for open strings. Its field contents is

- a gauge field A_μ with 8 on-shell d.o.f.,
- an adjoint real chiral spinor Ψ_m with 8 on-shell d.o.f..

It has a particularly simple action

$$S \sim \int d^{10}x \, \mathrm{tr}\big(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \gamma^\mu_{mn}\Psi^m D_\mu \Psi^n\big). \tag{11.6}$$

Supersymmetry acts as follows

$$\delta A^\mu \simeq \delta\epsilon^m \gamma^\mu_{mn}\Psi^n, \qquad \delta\Psi^m \simeq (\gamma^{\mu\nu})^m{}_n \delta\epsilon^n F_{\mu\nu}. \tag{11.7}$$

Supergravity Theories. Supergravity theories have a highly non-linear action, hence we shall only discuss some aspects of these theories. There are four relevant models for string theory:

- $\mathcal{N} = 1$ supergravity in 11D: M-theory.
- $\mathcal{N} = (1, 1)$ supergravity in 10D: type IIA supergravity.
- $\mathcal{N} = (2, 0)$ supergravity in 10D: type IIB supergravity.
- $\mathcal{N} = (1, 0)$ supergravity in 10D: type I supergravity.

The former three theories have $128 + 128$ d.o.f., the latter only $64 + 64$. Let us list the types of fields ($[n]$ refers to an n-form gauge field) in these theories along with the above gauge theory:

type	gr.	[4]	[3]	[2]	[1]	sc.	gravitini	spinors
M	1	-	1	-	-	-	1	-
IIA	1	-	1	1	1	1	(1,1)	(1,1)
IIB	1	1	-	2	-	2	(2,0)	(2,0)
I	1	-	-	1	-	1	(1,0)	(1,0)
SYM	-	-	-	-	1	-	-	(0,1)

(11.8)

[4] In this calculation the number of real dimensions matters, i.e. a complex dimension counts as two real dimensions.

[5] An analogous theory exists only in $D = 3, 4, 6$ due to particular spinor identities.

Note that M-theory has no 2-form field and no dilaton: It cannot be the low-energy limit of a string theory. Type IIA, IIB and I all have a 2-form field and a dilaton: they could arise as the low-energy limit of a string theory?!

11.2 Green–Schwarz Superstring

There are two approaches to formulate a supersymmetric string theory in $D = 10$ dimensions. We can either make the worldsheet or the target space supersymmetric. Both approaches turn out to be equivalent, and we shall begin with the latter, called the *Green–Schwarz* superstring.

Action. We shall discuss the type II superstring where we add two fermionic fields Θ_I^m, $I = 1, 2$, to the worldsheet theory. The fields transform as worldsheet scalars and target space spinors. They have equal or opposite chirality for so-called IIB and IIA string theory, respectively.

Target space is now a superspace with coordinates X^μ and Θ_I^m. The worldsheet theory of string theory is formulated in terms of line elements. The supersymmetric line elements Π receive some extra contributions from the fermionic directions

$$\Pi_\alpha^\mu = \partial_\alpha X^\mu + \delta^{IJ} \gamma^\mu_{mn} \Theta_I^m \partial_\alpha \Theta_J^n. \tag{11.9}$$

The action then takes the convenient form

$$\begin{aligned} S \sim & \int d^2\xi \sqrt{-\det g}\, g^{\alpha\beta} \eta_{\mu\nu} \Pi_\alpha^\mu \Pi_\beta^\nu \\ & + \int \Big(\big(\Theta^1 \gamma_\mu d\Theta^1 - \Theta^2 \gamma_\mu d\Theta^2\big)\, dX^\mu + \Theta^1 \gamma_\mu d\Theta^1 \Theta^2 \gamma^\mu d\Theta^2 \Big). \end{aligned} \tag{11.10}$$

In addition to diffeomorphisms, this action has so-called *kappa symmetry* which is a local worldsheet supersymmetry and which effectively removes half of the fermionic fields. Importantly, this symmetry applies only in $D = 10$ dimensions of target space!

Note: the fermions Θ have first and second class constraints. Non-linear equations of motion. In general difficult to quantise canonically. Conformal gauge does not resolve these difficulties.

Light Cone Gauge. It is convenient to apply light cone gauge which simplifies the model drastically: The action becomes quadratic such that the e.o.m. are linear

$$S \sim \int d^2\xi \left(\partial_{\mathrm{L}} \vec{X} \cdot \partial_{\mathrm{R}} \vec{X} + \tfrac{1}{2} \Theta_1 \cdot \partial_{\mathrm{R}} \Theta_1 + \tfrac{1}{2} \Theta_2 \cdot \partial_{\mathrm{L}} \Theta_2 \right). \tag{11.11}$$

The bosonic and fermionic d.o.f. are described slightly differently. The bosonic fields are exactly the same as for bosonic strings:

- They satisfy the second-order e.o.m. $\partial_{\mathrm{L}} \partial_{\mathrm{R}} \vec{X} = 0$.
- Both, the left and right moving d.o.f. are contained in the field $\vec{X}$ via $\partial_{\mathrm{L,R}} \vec{X}$.

- The fields transform in a vector representation $\mathbf{8}_\mathrm{v}$ of the transverse SO(8).

Conversely, the fermionic fields have the following features:

- They satisfy first-order equations $\partial_\mathrm{R}\Theta_1 = 0$ and $\partial_\mathrm{L}\Theta_2 = 0$.
- The left and right moving d.o.f. are contained in Θ_1 and Θ_2, respectively.
- The fields transform in real chiral spinor representations $\mathbf{8}_\mathrm{s}$ or $\mathbf{8}_\mathrm{c}$ of the transverse SO(8). They have equal or opposite chiralities for IIB or IIA, respectively: $\mathbf{8}_\mathrm{s} + \mathbf{8}_\mathrm{s}$ or $\mathbf{8}_\mathrm{s} + \mathbf{8}_\mathrm{c}$.

Spectrum. Next let us consider the vacuum energy, CFT central charge and anomaly cancellations:

- For the left and right movers, there are 8 bosonic and 8 fermionic d.o.f. each where the latter contribute with negative sign to the intercept a

$$a_{\mathrm{L/R}} = \tfrac{1}{2}8\zeta(-1) - \tfrac{1}{2}8\zeta(-1) = 0. \tag{11.12}$$

  There is no shift a for the L_0 constraint. Therefore the level zero is massless. There is no tachyon!
  Importantly, the number of bosonic and fermionic d.o.f. is precisely the same. This can only happen in particular low number of spacetime dimensions such as $D = 10$.
- Before light cone gauge, there are 10 bosonic fields X^μ and 32 fermionic fields Θ^m_I. Due to kappa symmetry, only half of the latter to the CFT central charge

$$c = 10 + 32\tfrac{1}{2} = 26. \tag{11.13}$$

  This number is the same as for bosonic string theory and cancels precisely against the contribution of ghost fields.
- The super-Poincaré anomaly cancels in light cone gauge.

As for the bosonic string, we can expand the closed superstring fields into Fourier modes. This leads to the bosonic modes α_n and fermionic modes β_n, where the modes $n < 0$, $n = 0$, $n > 0$ take the roles of creation operators, zero modes and annihilation operators, respectively.

The zero modes have a direct impact on the structure of the string vacuum states:

- The bosonic zero mode α_0 describes the centre of mass momentum: $\vec{q}$.
- The existence of fermionic zero modes β_0 implies the presence of a non-trivial supermultiplet at level zero. This supermultiplet consists of 8 bosonic and 8 fermionic states. Their representations depend on the type of fermionic zero modes:

$$\begin{aligned}
&\beta_0 \text{ chiral } (\mathbf{8}_\mathrm{s}): && \mathbf{8}_\mathrm{v} \leftrightarrow \mathbf{8}_\mathrm{c} && \text{vacuum} \to |\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{c}, q\rangle,\\
&\beta_0 \text{ anti-chiral } (\mathbf{8}_\mathrm{c}): && \mathbf{8}_\mathrm{v} \leftrightarrow \mathbf{8}_\mathrm{s} && \text{vacuum} \to |\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{s}, q\rangle.
\end{aligned} \tag{11.14}$$

The resulting string spectrum at level zero therefore depends on the types of zero modes

- Type IIA closed strings: $(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{s})\times(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c})$

$$\begin{aligned}\mathbf{8}_\mathrm{v}\times\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{s}\times\mathbf{8}_\mathrm{c}&=(\mathbf{35}_\mathrm{v}+\mathbf{28}_\mathrm{v}+\mathbf{1})+(\mathbf{56}_\mathrm{v}+\mathbf{8}_\mathrm{v}),\\ \mathbf{8}_\mathrm{v}\times\mathbf{8}_\mathrm{s}+\mathbf{8}_\mathrm{v}\times\mathbf{8}_\mathrm{c}&=(\mathbf{56}_\mathrm{s}+\mathbf{8}_\mathrm{c})+(\mathbf{56}_\mathrm{c}+\mathbf{8}_\mathrm{s}).\end{aligned}\tag{11.15}$$

  This is the particle spectrum of type IIA supergravity.
- Type IIB closed strings: $(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c})\times(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c})$

$$\begin{aligned}\mathbf{8}_\mathrm{v}\times\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c}\times\mathbf{8}_\mathrm{c}&=(\mathbf{35}_\mathrm{v}+\mathbf{28}_\mathrm{v}+\mathbf{1})+(\mathbf{35}_\mathrm{c}+\mathbf{28}_\mathrm{v}+\mathbf{1}),\\ \mathbf{8}_\mathrm{v}\times\mathbf{8}_\mathrm{s}+\mathbf{8}_\mathrm{v}\times\mathbf{8}_\mathrm{s}&=(\mathbf{56}_\mathrm{s}+\mathbf{8}_\mathrm{c})+(\mathbf{56}_\mathrm{s}+\mathbf{8}_\mathrm{c}).\end{aligned}\tag{11.16}$$

  This is the particle spectrum of type IIB supergravity.
- Type I closed strings arise as a $\mathbb{Z}_2$ projection of type IIB superstrings: $(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c})\times(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c})\bmod\mathbb{Z}_2$

$$(\mathbf{35}_\mathrm{v}+\mathbf{28}_\mathrm{v}+\mathbf{1})+(\mathbf{56}_\mathrm{s}+\mathbf{8}_\mathrm{c}).\tag{11.17}$$

  This is the particle spectrum of type I supergravity.
- Type I open strings: $\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c}$. This is the particle spectrum of $\mathcal{N}=1$ super Yang–Mills theory.

11.3 Ramond–Neveu–Schwarz Superstring

There is an alternative formulation for the superstring: the so-called Ramond–Neveu–Schwarz (RNS) superstring. This formulation has manifest worldsheet supersymmetry while spacetime supersymmetry is obscured.

Action. The action of the RNS superstring in conformal gauge consists of the real bosonic scalar fields X^μ and a pair of real fermionic spinors of either chirality $\Psi^\mu_{\mathrm{L,R}}$

$$S\sim\int d^2\xi\,\eta_{\mu\nu}\left(\tfrac{1}{2}\partial_\mathrm{L}X^\mu\partial_\mathrm{R}X^\nu+i\Psi^\mu_\mathrm{L}\partial_\mathrm{R}\Psi^\nu_\mathrm{L}+i\Psi^\mu_\mathrm{R}\partial_\mathrm{L}\Psi^\nu_\mathrm{R}\right).\tag{11.18}$$

This action is manifestly supersymmetric on the worldsheet which implies that bosons and fermions must transform equally under spacetime symmetries. Therefore also the fermionic fields are vectors of $\mathrm{SO}(9,1)$. This appears to violate the spin statistics theorem for the spacetime theory, but this problem can be resolved as we shall see shortly.

The bosonic fields behave as for the bosonic string. For the boundary conditions of the fermionic fields there is an important choice to be made; they can be either periodic or anti-periodic. This leads to two sectors of the theory which are analogous to the various open and closed string sectors. They are represented by two set of vacuum states which are unrelated by acting with string mode excitations.

Ramond Sector. One sector has periodic boundary conditions for the fermionic field

$$\Psi(\sigma + 2\pi) = +\Psi(\sigma). \tag{11.19}$$

It is called the *Ramond (R) sector*:

- Alike the bosonic modes α^μ_n, the fermionic fields are expanded into integer Fourier modes β^μ_n.
- Since there are fermionic zero modes β^μ_0, the vacuum is a supermultiplet. The states transform in a chiral and in an anti-chiral 16-component real spinor of $\mathrm{Spin}(9,1)$.
- The intercept is zero

$$a = -\tfrac{1}{2}8\zeta(-1) + \tfrac{1}{2}8\zeta(-1) = 0, \tag{11.20}$$

  hence there cannot be tachyons in this sector.

All above states transform as worldsheet spinors. Therefore they should respect fermionic statistics. However, there are states of either statistics in the Ramond sector.

This problem is resolved by the Glizzi–Scherk–Olive (GSO) projection which essentially restricts to states with fermionic statistics. This is a consistent projection of the string even in the presence of interactions. Note that this leaves a choice: One can assign the chiral vacuum states to be bosonic or fermionic. It leads to two inequivalent Ramond sectors R+ and R−.

Neveu–Schwarz Sector. An alternative consistent choice is periodic boundary conditions for the fermionic field

$$\Psi(\sigma + 2\pi) = -\Psi(\sigma). \tag{11.21}$$

This leads to the so-called *Neveu–Schwarz (NS) sector*.

- The fermionic fields are expanded into half-integer Fourier modes $\beta^\mu_{n+1/2}$.
- There are no fermionic zero modes therefore the string vacuum for this sector is a single fermionic scalar.
- The intercept is non-zero[6]

$$a = -\tfrac{1}{2}8\zeta(-1) + \tfrac{1}{2}8\zeta(-1,\tfrac{1}{2}) = \tfrac{1}{2}, \tag{11.22}$$

  hence the sector contains tachyons.

All the states transform as integer-spin representations of spacetime, but again there are states of either statistics. The GSO projection for the NS sector restricts to bosonic states, i.e. states with an odd number of fermionic generators β. The GSO projection has the beneficial effect of removing the tachyonic level 0. Physical states start at level 1/2 which is massless.

[6]The regularised contribution of the half-integer fermionic modes is captured by a generalised zeta function $\zeta(-1,\tfrac{1}{2}) = 1/24$.

Superstring Models. The fermion periodicity conditions can be selected individually for the left and right moving fields. There are four sectors in a consistent formulation of the closed superstring: NS-NS, NS-R, R-NS, R-R. The NS-NS and R-R sectors provide the bosonic particles, the NS-R and R-NS sectors the fermionic ones.

For each R sector there is a choice of GSO projection. For equal or opposite choices for the left and right moving sectors, one obtains the type IIB and IIA superstrings, respectively.

The type I closed superstring is obtained by a further projection. Effectively, the R-R sector at the massless level is eliminated.

In type I open superstrings the left and right moving sectors are related, therefore there is only one NS and one R sector.

Superconformal Algebra. Conformal symmetry combines with the manifest supersymmetry on the worldsheet to superconformal symmetry. As usual, there are two copies of conformal symmetry for a closed string: one for the left movers and one for the right movers. Let us consider just one copy.

The stress-energy tensor and conformal supercurrent read:

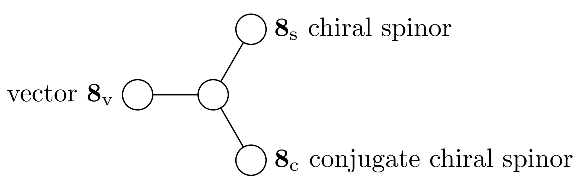
$$T = \partial X \cdot \partial X + \tfrac{i}{2}\Psi \cdot \partial\Psi, \qquad J = \Psi \cdot \partial X. \tag{11.23}$$

The superconformal algebra is generated by bosonic generators L_n and fermionic generators G_r. They obey the following algebra[7]

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{1}{8}cm(m^2-1)\delta_{m+n},\\
[L_m, G_r] &= (\tfrac{1}{2}m - r)G_{m+r},\\
\{G_r, G_s\} &= 2L_{r+s} + \tfrac{1}{2}c(r^2 - \tfrac{1}{4})\delta_{r+s}.
\end{aligned} \tag{11.24}$$

Here the bosonic generators L_n occupy integer modes n while the modes of the fermionic generators depend on the sector. In the R or NS sectors, $2r$ is even or odd, respectively. The corresponding superalgebras are called Ramond and Neveu–Schwarz algebras.

Comparison. The GS and RNS approaches to superstrings yield the same results. When both models are restricted to light cone gauge one finds that they are related by a special feature of the group SO(8): the Dynkin diagram has a three-fold symmetry called *triality*.



(11.25)

[7]Note that the central charge is $c = D$ with a conventional factor of $3/2$ w.r.t. the definition of the bosonic Virasoro algebra.

It relates the vector and both chiral spinor representations. In this way one can change the representations of the string fields without changing the spectrum.

Let us compare some characteristic features of the two approaches:

	GS	RNS
fermions are spinors in	target space	worldsheet
worldsheet supersymmetry	(yes)	manifest
superconformal field theory	no	yes
target space supersymmetry	manifest	(yes)
supergravity couplings	all	some (NS-NS)
spacetime covariant	no	(yes)

(11.26)

In fact, a third approach exists: The *pure spinor* formulation introduced by Berkovits. Here one introduces auxiliary bosonic spinor fields λ satisfying the non-linear constraint $\lambda\gamma^\mu\lambda = 0$. It shares several benefits of the GS and RNS formulations.

11.4 Branes

Let us now discuss D-branes. We have learned that the ends of open strings couple to D-branes and that the open string spectrum carries the fluctuations of D-branes. Hence in superstring theory D-branes should receive fermionic degrees of freedom, let us therefore inspect the spectrum of open strings coupled to a Dp-brane:

- The massless modes are described by $\mathcal{N} = 1$ super Yang–Mills theory reduced to $(p+1)$ dimensions.[8]
- The spectrum has a tower of heavy string modes.
- A scalar tachyon may be present depending on the particular situation.

Stable Dp-Branes. Some D-branes are stable others are not. The presence of an open string tachyon indicates instability of the D-brane:

- D-branes in bosonic string theory are always unstable.
- Dp-branes for IIB superstring are stable for p odd.
- Dp-branes for IIA superstring are stable for p even.
- Since T-duality changes the dimension of D-branes by one unit, it must also may map between type IIA and IIB theories.

Stability is related to supersymmetry. The boundary conditions break some of the symmetry:

- Lorentz symmetry: $\mathrm{SO}(9,1) \to \mathrm{SO}(p,1) \times \mathrm{SO}(9-p)$.
- 16 supersymmetries preserved for p odd/even in IIB/IIA.
- no supersymmetries preserved for p even/odd in IIB/IIA.

Supersymmetry removes the open string tachyon and therefore stabilises open strings and particular D-branes.

[8]This theory has a vector field, 16 fermionic spinor fields (in total) and $9-p$ scalar degrees of freedom.

Supergravity p-Branes. D-branes are non-perturbative objects, they are not seen perturbatively due to their large mass. Stable Dp-branes have a low-energy limit as solutions to the supergravity equations of motion, so-called p-branes.

Alike the fundamental string and magnetic brane solutions, p-branes are supported by a $(p+1)$-form, gravity and the dilaton field.

- Both type II string theories have the dilaton and a two-form field from the NS-NS sector.
- In addition IIB and IIA strings have forms of even or odd degree, respectively, from the R-R sector. They are relevant for the stable Dp-branes.

The stable p-branes enjoy a several useful features:

- A p-brane carries $(p+1)$-form charge. The charge prevents the p-branes from evaporating.
- The charge density is proportional to the mass density.
- 16/32 supersymmetries preserved: This is a so-called *1/2 BPS* condition.
- There is a non-renormalisation theorem for 1/2 BPS objects: p-branes are independent of the coupling strength, they are the same at weak/intermediate/strong coupling. Therefore half BPS p-branes describe Dp-branes exactly.

11.5 Other Superstrings

There are three further superstring theories which we shall briefly discuss.

Type-I Superstring. Consider now superstrings ending on spacetime-filling D9-branes. The cancellation of all gravity and gauge anomalies requires:

- a gauge group of dimension 496,
- some special property of the charge lattice.

There are only two solutions: SO(32) and $E_8 \times E_8$. Here the gauge group must be SO(32). This breaks half of the supersymmetry and yields the type I superstring. This model is sometimes considered an independent string theory. Alternatively it may be viewed as IIB string theory with 16 D9 branes and a spacetime-filling orientifold plane. As such it is part of IIB string theory.

Heterotic Superstrings. We have seen that there is almost no interaction between the left and right movers. Let us exploit this fact:

- construct the left moving sector as for the superstring: $D = 10$ bosonic dimensions plus fermions.
- set up the right moving sector as for bosonic string: 26 bosonic fields which amount to $D = 10$ dimensions plus 16 extra bosonic degrees of freedom.

This string is called the *heterotic string*. It has 16 supersymmetries from the left moving sector.

Again, the anomaly cancellation requires a particular gauge symmetry:

- HET-O: SO(32) or
- HET-E: $E_8 \times E_8$.

The gauge group is supported by the 16 additional internal degrees of freedom.

In particular, the HET-E model is interesting because E_8 contains several potential GUT groups:

$$E_5 = SO(10), \qquad E_4 = SU(5), \qquad E_3 = SU(3) \times SU(2). \tag{11.27}$$

12 Effective Field Theory

In the previous chapters, the main concepts of string theory have been introduced: worldsheet and target space, conformal field theory, supersymmetry, string scattering as well as the spectrum of different types of theories. In what follows, string theory shall be connected to field theories in various dimensions.

12.1 Effective Action and Compactifications

Let us begin the current chapter by reviewing the two main methods relating string theories to field theories: the low-energy expansion and compactification. Both concepts have appeared in previous chapters in several contexts, so the discussion here will be rather brief.

Effective Action. Effective actions in the context of quantum field theory allow to access the full dynamics of the theory including loop effects in principle. Practically, those actions are usually known up to a certain loop order. For string theories, the concept remains similar: corrections originating in the stringy nature of the states as well as string loop corrections are condensed into one field theory action. Furthermore, string effective actions are available for theories in different numbers of dimensions and thus incorporate the effects of compactification. A further restriction could be to limit the attention to a part of the spectrum or to a certain regime of energies: there are effective actions for the bosonic particles in a theory as well as actions reproducing just a subset of the scattering amplitudes. The most prominent examples of effective actions for string theories are the *low-energy effective actions* which are obtained by considering the point-particle limit $\alpha' \to 0$ and thereby projecting on the massless spectrum of the string theory. By construction, those actions do not represent the dynamics and the particle content of the full theory: they are valid only in the regime they are designed for. A well-studied example of an effective action is the Dirac–Born–Infeld action.

Compactification. While the principles of compactification have been discussed before, it shall be applied in order to obtain different limits of string theories here. The compactified theory depends on the geometry of the compactification manifold sensitively as the fields from the compactified dimensions appear as effective fields in the compactified theory along with modified interactions, masses etc.. Correspondingly, choosing a suitable compactification manifold is an efficient (and the only) way to shape the compactified effective theory. The easiest way of removing the unwanted extra dimensions – the torus compactification – is not sufficient to yield phenomenologically interesting models; instead we will have to consider more complicated manifolds, in order to engineer (parts of) the models proven valid in the experiments in our world.

In performing the compactification one has to be careful with the signature: while everything works straightforwardly on a torus with Euclidean signature, a compactification on Minkowskian manifolds is more difficult and leads to scalars of negative norm or non-compact internal symmetry of the resulting theory. In order to avoid those difficulties, we will assume to have complex-valued fields throughout the discussion of compactification and effective actions. For a theory with Minkowskian signature one can impose suitable reality conditions for the desired spacetime signature later on.

Derivation of Effective Actions for String Theories. Given a full string theory defined by its worldsheet action, it is not straightforward to immediately write down an effective low-energy target space action for a particular set of states. In order to obtain it, one would usually perform the following steps:

- Calculate several scattering amplitudes in the compactified version of the original string theory for the part of the spectrum which shall be described by the effective action.
- String theory exhibits two parameters: the inverse string tension $\alpha' = \kappa^2$ and the string coupling g_{s}. In order to obtain an effective *field* theory, one has to remove one of those coupling constants. This can be done by either considering only α' and thus limit the attention to e.g. tree-level string amplitudes or by uniting the parameters into a new effective parameter.
- Consider the symmetry of the amplitudes and find an action in the correct dimension exhibiting these symmetries which reproduces the scattering amplitudes.
- The effective theory might carry only a subset of the symmetries present in the original theory.
- In general, deriving an effective action is not an easy task. However, many effective actions for string theories are known, at least to leading order in the coupling constant.

Considering tree-level string amplitudes, the torus compactification to four dimensions yield the actions of $\mathcal{N} = 4$ super Yang–Mills theory (from type I open string theory) and $\mathcal{N} = 8$ supergravity (from type II closed string theory) as their leading terms in the effective actions. For those theories, several corrections organised in powers of α' are known.

A huge variety of possible compactifications exist. Except the two examples above, many other known low-energy effective actions start with well-known field theory actions. However, the goal of finding a compactification leading to an effective model reproducing the standard model coupled to gravity have not been successful so far.

12.2 Open Strings

Yang–Mills field theories are the backbone of gauge theories describing our world: several of them can be combined to yield the standard model. Supersymmetric versions of Yang–Mills theories can be obtained from the low-energy limit of open string theories. Although unrealistic, those supersymmetric Yang–Mills theories serve as prototypes and share many properties with their non-supersymmetric relatives.

Since the ends of open strings couple to D-branes, the open string spectrum carries their fluctuations. As we are dealing with a supersymmetric theory, the D-brane is stable and there is no tachyon in the spectrum. The massless modes of those fluctuations are described by $\mathcal{N}=1$ super Yang–Mills theory reduced to $(p+1)$ dimensions for a Dp-brane.

The $\mathcal{N}=1$ SYM theory in $D=10$ has already been introduced in the previous chapter:

$$S \sim \int d^{10}x\,\mathrm{tr}\big(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \gamma^{\mu}_{mn}\Psi^m D_\mu \Psi^n\big). \tag{12.1}$$

It has a vector field, a total of 16 fermionic spinor fields and $9-p$ scalar degrees of freedom. By compactifying the $\mathcal{N}=1$ SYM theory on the torus, the number of supercharges remains the same. Considering the available representations of spinors in different dimensions, one will find the following Yang–Mills theories to exist:

dimension	spinor dim.	theory
10	16	$\mathcal{N}=1$ SYM
6	8	$\mathcal{N}=2$ SYM
4	4	$\mathcal{N}=4$ SYM
3	2	$\mathcal{N}=8$ SYM

(12.2)

Since we are interested in four dimensions for phenomenological reasons, let us consider $\mathcal{N}=4$ SYM theory more precisely.

$D=4$, $\mathcal{N}=4$ SYM Theory. The theory is the maximally supersymmetric theory for spin-1 particles in four dimensions. It is of particular interest as a toy model for phenomenologically more interesting theories. States are organized in one irreducible $\mathcal{N}=4$ multiplet consisting of 16 states: 2 gluons, 8 gluinos and 6 real scalars. The ten-dimensional Lorentz symmetry splits into $\mathrm{SO}(3,1)\times\mathrm{SO}(6)$, which in turn is contained in the complete symmetry group of the theory, $\mathrm{PSU}(2,2|4)$.

While $\mathcal{N}=4$ SYM can be obtained from compactifying the $\mathcal{N}=1$ SYM theory on a torus, one can as well compactify type I string theory on the same manifold. In this situation, $\mathcal{N}=4$ SYM theory appears as the leading order in the low-energy effective action accompanied by corrections of higher order in $\alpha'=\kappa^2$ originating in string theory. In order to discuss those corrections, let us for simplicity consider the gluon sector of the theory only. In this sector, the effective action reads

$$S \sim \int d^4x\Big(-\tfrac{1}{4}\,\mathrm{tr}(F^2) - \alpha'^2\zeta_2\,\mathrm{tr}(F^4) + \mathcal{O}(\alpha'^3)\Big), \tag{12.3}$$

where

$$F^4 = F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\sigma F_\sigma{}^\mu + 2F_\mu{}^\nu F_\rho{}^\sigma F_\nu{}^\rho F_\sigma{}^\mu \\ - \tfrac{1}{4}F_{\mu\nu}F_{\rho\sigma}F^{\mu\nu}F^{\rho\sigma} - \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \,. \tag{12.4}$$

The particular linear combination of different contractions of indices for the F^4-term is the only combination allowed by supersymmetry. Supersymmetry is as well responsible for the absence of the first-order correction: the supersymmetric extension of the term $\alpha'\pi \operatorname{tr} F^3$ implies sets amplitudes which are not compatible with supersymmetric Ward-identities.

12.3 Closed Strings

Type IIA and IIB Supergravity. Let us start the discussion of effective actions for the closed string with the two $\mathcal{N}=2$ supergravities in ten dimensions: type IIA and type IIB. As noted in the previous chapter, Dp-branes for the superstring are stable for odd p in type IIA theories and stable for even p in type IIB theories. This nicely fits with the fact that T-duality changes the dimension of D-branes by one unit.

As discussed in the last chapter, supergravity p-branes are supported by a $(p+1)$-form, gravity, antisymmetric tensor and the dilaton field. While the latter three originate in the NS-NS sector, the $(p+1)$-forms are in the R-R sector. Having only odd p-branes in the type IIA theory, there will be even forms in this theory exclusively. As the fermionic states of the two theories are complicated, let us stick here with the bosonic ones and collect them for type IIA and type IIB theory in the following table:

	sector	fields	field strengths
type IIA	NS-NS	$G_{\mu\nu}$, Φ, $B_{\mu\nu}$	$H_{\mu\nu\lambda}$
	R$^+$-R$^-$	C_1, C_3	F_2, F_4
type IIB	NS-NS	$G_{\mu\nu}$, Φ, $B_{\mu\nu}$	$H_{\mu\nu\lambda}$
	R$^+$-R$^+$	C_0, C_2, C_3	F_1, F_3, F_5

(12.5)

For both theories, the massless bosonic spectrum in the NS-NS sector is identical to the one in purely bosonic string theory. Thus the action reads

$$S_{\rm NS} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-\det G}\,e^{-2\Phi} \\ \cdot \big[R - \tfrac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\,\partial^\mu\Phi + \mathcal{O}(\alpha')\big]. \tag{12.6}$$

Different bosonic degrees of freedom from the R-R sector will result in different effective actions, however. Conveniently these actions are expressed in terms of the field strengths $F_{i+1} = dC_i$. They split into a term coupling to the metric $S_{\rm RR}$ as well as a purely topological term $S_{\rm CS}$. For type IIA theory the action reads

$$S_{\rm RR} = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\,\sqrt{-\det G}\,\big(F_2^2 + \tilde{F}_4^2\big), \\ S_{\rm CS} = -\frac{1}{4\kappa_{10}^2}\int B\wedge F_4\wedge F_4, \tag{12.7}$$

where we set for simplicity $\tilde{F}_4 = F_4 - C_1 \wedge H_3$. For type IIB, one finds

$$S_{\text{RR}} = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-\det G} \left(F_1^2 + \tilde{F}_3^2 + \tfrac{1}{2}\tilde{F}_5^2\right),$$
$$S_{\text{CS}} = -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3, \tag{12.8}$$

where $\tilde{F}_3 = F_3 - C_0 \wedge H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. Deriving the equations of motion and the Bianchi identities for the field $\tilde{F}_5$ yields

$$d{*}\tilde{F}_5 = d\tilde{F}_5 = H_3 \wedge F_3, \tag{12.9}$$

which is compatible with the self-duality of the five-form in ten dimensions: ${*}\tilde{F}_5 = \tilde{F}_5$. Note, however, that this condition is not implied by the action and has to be imposed separately.

***D* = 4, 𝒩 = 8 Supergravity.** The maximally supersymmetric theory for particles of spin 2 in four dimensions contains one irreducible $\mathcal{N} = 8$ multiplet consisting of 256 states: 2 gravitons, 16 gravitini, 56 vector particles, 112 spin-1/2 particles and 70 real scalars. Similar to the situation in $\mathcal{N} = 4$ SYM above, $\mathcal{N} = 8$ supergravity is corrected by string theory effects in the effective action. In the graviton sector, the effective action reads

$$S \sim \int d^4x \sqrt{-\det G} \left(R + \alpha'^3 \zeta_3 R^4 + \mathcal{O}(\alpha'^4)\right), \tag{12.10}$$

where R^4 is a contraction of Riemann tensors dictated by supersymmetry. Again, as for $\mathcal{N} = 4$ SYM, the absence of the first-order and second-order corrections is caused by supersymmetry. There is one peculiarity: there is another supergravity theory, which naively does not seem to correspond to a string theory.

***D* = 11 Supergravity.** As discussed above, there is the 32-dimensional Majorana representation for spinors available in 11 dimensions. The corresponding supergravity theory will thus have $\mathcal{N} = 1$ supersymmetry. Considering the representations of the little group SO(9) implied by the spinor representations, one finds the massless spectrum to consist of two bosonic fields, the metric $G_{\mu\nu}$, a 3-form potential $A_{\mu\nu\rho}$ as well as a spin-3/2 fermion, the gravitino. The supersymmetry algebra is identical to the one appearing in ten-dimensional type IIA theory. This is not a coincidence as will be pointed out in the next chapter.

12.4 Relations Between String Amplitudes

The observables of a string theory are the scattering amplitudes. While the calculation of those objects has been explained before, one can obtain additional information by considering their worldsheet origin: beyond the obvious cyclicity and reflection symmetry one can – using complex analysis – relate further worldsheet calculations and thus derive relations between the corresponding amplitudes. There are two distinct approaches: combining two open-string worldsheets into a closed one will lead to the Kawai–Lewellen–Tye (KLT) relations. Choosing different contours for the evaluation of an open-string amplitude will lead to the monodromy relations between those. Nicely, both relations survive the low-energy limits and thus carry over to the amplitudes in corresponding four-dimensional field theories.

KLT Relations. The KLT relations relate scattering amplitudes in closed string theories to the ones in open string theories. This is not a duality: the two physical systems described by open and closed strings have a different spectrum and are distinct.

Any vertex operator corresponding to a massless state in closed string theory can be written as product of two vertex operators in open string theory:

$$V^{\text{closed}}(z_i, \bar{z}_i) = V^{\text{open}}_{\text{left}}(z_i)\, \bar{V}^{\text{open}}_{\text{right}}(\bar{z}_i)\,. \tag{12.11}$$

While in the closed string the insertion points z_i, $\bar{z}_i$ are integrated over a two-sphere, in the open-string case the real z_i are integrated over the boundary of a disk. In order to express the closed-string integral in terms of open-string integrals, one has to identify and relate the contours of integration in the open-string integrals in a way which yields a consistent closed-string expression. Deforming the contours in the open string integral one will yield various phase factors.

After doing so, one can express the closed-string amplitudes in terms of a sum of products of open-string amplitudes. The total permutation symmetry of the insertion points on the sphere (and thus the total permutation symmetry of the external legs in the closed-string integral) is ensured by a taking particular sums of different products of open-string amplitudes with permuted legs.

KLT relations are a very convenient way to calculate amplitudes in closed string theory and more so in their low-energy limits, supergravity. In fact: many calculations in supergravity could not have been performed without KLT relations.

The explicit relations depend on the external states of the amplitudes. For the example of pure tachyon amplitudes one finds

$$M^{\text{closed}}_{4,\text{tach}}(s,t) \sim \frac{1}{\pi} \sin(\alpha' \pi t/4) A^{\text{open}}_{4,\text{tach}}(s/4, t/4) A^{\text{open}}_{4,\text{tach}}(t/4, u/4)\,. \tag{12.12}$$

For external gluons, the four-point and five-point KLT relations read

$$\begin{aligned}
M^{\text{closed}}_{4,\text{grav}}(1,2,3,4) &= \frac{-i}{\alpha' \pi} \sin(\alpha' \pi s) \\
&\quad \cdot\; A^{\text{open}}_{4,\text{gluon}}(1,2,3,4)\, A^{\text{open}}_{4,\text{gluon}}(1,2,4,3)\,, \\
M^{\text{closed}}_{5,\text{grav}}(1,2,3,4,5) &= \frac{i}{\alpha'^2 \pi^2} \sin(\alpha' \pi s_{12}) \sin(\alpha' \pi s_{34}) \\
&\quad \cdot A^{\text{open}}_{5,\text{gluon}}(1,2,3,4,5)\, A^{\text{open}}_{5,\text{gluon}}(2,1,4,3,5) \\
&\quad + \frac{i}{\alpha'^2 \pi^2} \sin(\alpha' \pi s_{13}) \sin(\alpha' \pi s_{24})
\end{aligned}$$

$$\cdot\, A^{\text{open}}_{5,\text{gluon}}(1,3,2,4,5)\, A^{\text{open}}_{5,\text{gluon}}(3,1,4,2,5)\,. \tag{12.13}$$

Here Mandelstam variables are defined as $s_{ij} = -(q_i + q_j)^2$, where for the four-point case the usual $s = s_{12}$, $t = s_{14}$ and $u = s_{13}$ have been used.

The above equalities are exact relations between string theory amplitudes, and so they are valid order by order in α'. For example, in the limit $\alpha' \to 0$ KLT will relate gluon amplitudes in $\mathcal{N} = 4$ SYM to graviton amplitudes in $\mathcal{N} = 8$ supergravity. Being an exact relation between open and closed string amplitudes, the KLT relations are valid for each order in α' individually. In particular, one can relate the string corrections to $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity: using KLT relations one can for example show that the absence of the first-order and second-order corrections to supergravity amplitudes is implied by the particular form of the string corrections to the $\mathcal{N} = 4$ SYM theory.

KLT relations do not only relate gluon and graviton amplitudes in the field theory limits of open and closed string theories, but amplitudes from the full multiplets. In particular, the tensor-decomposition of the Fock space reads

$$[\mathcal{N} = 8] \;\longleftrightarrow\; [\mathcal{N} = 4]_{\text{L}} \otimes [\mathcal{N} = 4]_{\text{R}}. \tag{12.14}$$

Monodromy Relations. Monodromy relations arise from deforming the contour of integration and thus relating open-string amplitudes with different ordering of legs (or different successions of inserting the corresponding vertex operators on the boundary of the disk. The simplest example is again the four-point amplitude. Here we will not specify the particle content, but rather focus on the kinematical dependence of the amplitude. Fixing the Möbius symmetry on the worldsheet by choosing $z_1 = 0$, $z_3 = 1$ and $z_4 = \infty$, one can write the three configurations of amplitudes which are not related by cyclicity and reflection as

$$\begin{aligned}
A^{\text{open}}_4(2,1,3,4) &\sim \int_{-\infty}^{0} dz_2\,(-z_2)^{\alpha' q_1\cdot q_2}(1-z_2)^{\alpha' q_2\cdot q_3},\\
A^{\text{open}}_4(1,2,3,4) &\sim \int_{0}^{1} dz_2\,(z_2)^{\alpha' q_1\cdot q_2}(1-z_2)^{\alpha' q_2\cdot q_3},\\
A^{\text{open}}_4(1,3,2,4) &\sim \int_{1}^{\infty} dz_2\,(z_2)^{\alpha' q_1\cdot q_2}(z_2-1)^{\alpha' q_2\cdot q_3}\,.
\end{aligned} \tag{12.15}$$

Analytically continuing the variable z_2 to the complex plane one can consider integrating the integrand of the first amplitude, $A^{\text{open}}_4(2,1,3,4)$ over a contour closed at infinity. Assuming the poles at $z_2 = 0$ and $z_2 = 1$ to be outside the integration regime, one finds immediately

$$\begin{aligned}
0 = &\int_{-\infty}^{0} dz_2\,(-z_2)^{\alpha' q_1\cdot q_2}(1-z_2)^{\alpha' q_2\cdot q_3}\\
&+ \int_{0}^{1} dz_2\,(-z_2)^{\alpha' q_1\cdot q_2}(1-z_2)^{\alpha' q_2\cdot q_3}\\
&+ \int_{0}^{\infty} dz_2\,(-z_2)^{\alpha' q_1\cdot q_2}(1-z_2)^{\alpha' q_2\cdot q_3},
\end{aligned} \tag{12.16}$$

which can be easily rewritten as

$$\begin{aligned}
&A_4^{\text{open}}(2,1,3,4)\\
&= -\int_0^1 dz_2\,(-z_2)^{\alpha' q_1\cdot q_2}(1-z_2)^{\alpha' q_2\cdot q_3}\\
&\quad -\int_0^\infty dz_2(-z_2)^{\alpha' q_1\cdot q_2}(1-z_2)^{\alpha' q_2\cdot q_3}\\
&= -e^{i\pi\alpha' q_1\cdot q_2}\int_0^1 dz_2\,(z_2)^{\alpha' q_1\cdot q_2}(1-z_2)^{\alpha' q_2\cdot q_3}\\
&\quad - e^{i\pi\alpha'(q_1\cdot q_2+q_2\cdot q_3)}\int_0^\infty dz_2\,(z_2)^{\alpha' q_1\cdot q_2}(z_2-1)^{\alpha' q_2\cdot q_3}\\
&= -e^{i\pi\alpha' q_1\cdot q_2}A_4^{\text{open}}(1,2,3,4) - e^{i\pi\alpha'(q_1\cdot q_2+q_2\cdot q_3)}A_4^{\text{open}}(1,3,2,4)\,. \qquad (12.17)
\end{aligned}$$

The above equality is the monodromy relation for the four point string amplitudes. In the low-energy limit ($\alpha' \to 0$), its real part corresponds to the photon-decoupling identity for gauge theory amplitudes, while the imaginary part yields the Bern–Carrasco–Johansson relations. The analysis can be performed in the same way for amplitudes with more external legs, which leads to the general form of the monodromy relations:

$$\begin{aligned}
&A_n^{\text{open}}(1,2,3,4,\ldots,n) + e^{i\alpha'\pi s_{12}}A_n^{\text{open}}(2,1,3,4,\ldots,n)\\
&\quad + e^{i\alpha'\pi(s_{12}+s_{13})}A_n^{\text{open}}(2,3,1,4,\ldots,n)\\
&\quad + \ldots\\
&\quad + e^{i\alpha'\pi(s_{12}+s_{13}+\ldots+s_{1,n-1})}A_n^{\text{open}}(2,3,4,\ldots,n-1,1,n) = 0, \qquad (12.18)
\end{aligned}$$

where again $s_{ij} = -(q_i+q_j)^2$. Employing cyclicity, reflection symmetry as well as the monodromy relations collectively reduces the number of independent amplitudes with n legs to $(n-3)!$.

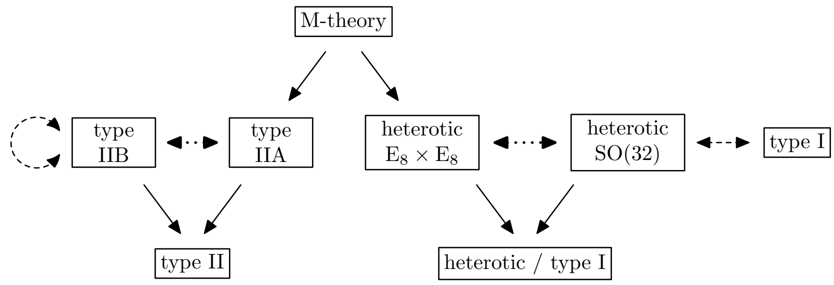
13 String Dualities

In the early 90's five different string theories were defined at the perturbative level only, while there was little understanding of the dynamical principles of the theory. Considering the degenerate ground states after compactification (which are parametrised by scalars/moduli), it was not clear which of the reductions could possibly correspond to the Standard Model. Most annoyingly, there were five different versions of string theory available while the theory had set off to be a candidate for a unique unifying theory. The resolution to those problems appeared with the advent of string dualities.

Two theories are called dual, if they describe the same physics using different "languages", that is, different fields, coupling constants and interactions. Hereby dualities can relate fields of completely different nature. Dualities lead to the identification of different vacua and finally allow – in combination with weak-coupling or strong-coupling limits – to identify M-theory as the link between the five different string theories, thus being closer to the unified description aimed at.

While T-duality will be the main working example here, there are other forms of dualities, in particular S-duality. In contrast to T-duality, which is a weak-weak duality relating the weak-coupling regimes of two theories to each other, S-duality is more interesting from the point of understanding the complete dynamics of a theory: It maps the non-perturbative strong-coupling sector of one theory to the perturbative weak-coupling sector of another theory.

The goal of this chapter is to explain the arrows in the following figure:



(13.1)

Here, dashed arrows denote S-duality, dotted arrows mark T-duality and solid arrows are compactifications on a suitable interval.

13.1 T-Duality

Within the context of the string theories explored so far there are two famous examples of T-duality: type IIA and IIB string theory can be shown to describe the same physics if one dimension is compactified on a circle. This is the example to be explored below. Furthermore, the two heterotic string theories HET-O and HET-E with gauge groups SO(32) and $E_8 \times E_8$, respectively, are related by T-duality.

T-Duality between IIA and IIB String Theory. Consider T-duality between the background fields in the NS-NS and the R-R sector: G, B, Φ, F. The relations between the fields G and B from the NS-NS sector and their duals $\tilde{G}$ and $\tilde{B}$ can be obtained by starting from the non-linear sigma-model in conformal gauge

$$S = -\frac{1}{4\pi\kappa^2} \int d^2\xi \left(\eta^{\alpha\beta}\, G_{\mu\nu} - \varepsilon^{\alpha\beta}\, B_{\mu\nu}\right) \partial_\alpha X^\mu\, \partial_\beta X^\nu\,. \tag{13.2}$$

In order to derive the so-called Buscher rules for the fields G and B, the ten-dimensional metric and antisymmetric tensor need to be split in Kaluza–Klein form for a compact direction X^9. After doing so and requiring the background fields G and B to be independent of the compact dimension, one finds that the fields are related by:

$$\begin{aligned} \tilde{G}_{ij} &= G_{ij} - \frac{G_{i9}G_{j9} - B_{i9}B_{j9}}{G_{99}}\,, & \tilde{G}_{9i} &= -\frac{B_{9i}}{G_{99}}\,, & \tilde{G}_{99} &= \frac{1}{G_{99}}\,, \\ \tilde{B}_{ij} &= B_{ij} + \frac{G_{i9}B_{j9} - G_{j9}B_{i9}}{G_{99}}\,, & \tilde{B}_{9i} &= -\frac{G_{9i}}{G_{99}}\,. \end{aligned} \tag{13.3}$$

The dilaton transformation cannot be derived that easily: it can be inferred from demanding conformal invariance of the full (non-gauged) non-linear sigma-model action

$$\begin{aligned} S = -\frac{1}{4\pi\kappa^2} \int d^2\xi \Big[&\left(\sqrt{-\det g}\, g^{\alpha\beta}\, G_{\mu\nu} - \varepsilon^{\alpha\beta}\, B_{\mu\nu}\right) \partial_\alpha X^\mu\, \partial_\beta X^\nu \\ &+ \kappa^2 \sqrt{-\det g}\, \Phi(X)\, R[g] \Big]\,. \end{aligned} \tag{13.4}$$

T-duality for the fields from the R-R sector can be derived in a similar manner. It is easy to check that the number of components for the R-R fields F_i in type IIA string theory equals the number of free components in the type IIB theory. Indeed one finds

$$\begin{aligned} &\underbrace{\binom{10}{0}}_{F_0} + \underbrace{\binom{10}{2}}_{F_2} + \underbrace{\binom{10}{4}}_{F_4} = 1 + 45 + 210 = 256, \\ &\underbrace{\binom{10}{1}}_{F_1} + \underbrace{\binom{10}{3}}_{F_3} + \underbrace{\frac{1}{2}\binom{10}{5}}_{F_5} = 10 + 120 + 126 = 256. \end{aligned} \tag{13.5}$$

where the R-R scalar F_0 as well as a factor of 1/2 taking care of the self duality of F_5 have been taken into account. In open string theory, strings are tied to D-branes by their boundary conditions. As T-duality interchanges Dirichlet and Neumann boundary conditions, it is consistent to find D-branes of even dimensions in type IIA string theory, while type IIB contains D-branes of odd dimension.

Accordingly, T-duality has to relate even and odd field strengths. In a convenient gauge, the T-duality rules for the R-R fields read

$$
\begin{aligned}
F_{n,\,9i_2\ldots i_n} &= -F_{(n-1),\,i_2\ldots i_n} + (n-1)G_{99}^{-1}G_{9[i_2}F_{(n-1),\,9i_3\ldots i_n]}\,, \\
F_{n,\,i_1i_2\ldots i_n} &= -F_{(n+1),\,9i_1\ldots i_n} + nB_{9[i_1}F_{n,\,9i_2\ldots i_n]}\,.
\end{aligned}
\tag{13.6}
$$

Both transformations are *not* a symmetry of the action.

T-duality toggles the chirality of the fermions: while type IIA is non-chiral, type IIB is a chiral theory. Upon reduction on a circle to nine dimensions, the two theories coincide precisely, on the level of the action. Thus, the IIA and IIB theories in ten non-compact dimensions are different limits of the same theory in nine non-compact dimensions: one time with an infinitely large tenth dimension, one time with a small tenth dimension. In particular one finds

$$
\frac{R_A}{\kappa} = \sqrt{G_{99}} = \frac{1}{\sqrt{\tilde{G}_{99}}} = \frac{\kappa}{R_B}\,. \tag{13.7}
$$

T-Duality for Toroidal Compactifications. Instead of compactifying just one dimension, let us now consider a compactification of several dimension on a d-dimensional torus T^d. In each compact dimension with index $k \in \{D-d, \ldots, D-1\}$ one imposes

$$
X^k = X^k + 2\pi w^k\,, \tag{13.8}
$$

where w^k is the winding number for the compact dimension k. Different geometries are conveniently described by a compactification lattice Λ^d such that

$$
T^d = \mathbb{R}^d \big/ \Lambda^d. \tag{13.9}
$$

The information on the geometry of the torus is contained in the metric of the compact dimensions. The basis vectors e_i of the compactification lattice Λ^d and those for the dual lattice Λ^{*d}, e^{*i}, can be related to the metric in the following way:

$$
\sum_{a=1}^{d} e_k^a e_l^a = 2\pi G_{kl}, \quad \sum_{a=1}^{d} e_k^a e_a^{*l} = \delta_k^l, \quad \sum_{a=1}^{d} e_a^{*k} e_a^{*l} = \frac{1}{2\pi}(G^{-1})^{kl}. \tag{13.10}
$$

The quantisation of the momenta and the possibility for the string to wrap around each of the compact dimensions yield expressions for the left and right moving centre-of-mass momenta. In terms of the basis vectors of the dual lattice they can be expressed as:

$$
\begin{aligned}
p_{Ra} &= \big[n_a + w^l(B-G)_{kl}\big]e_a^{*k}, \\
p_{La} &= \big[n_a + w^l(B+G)_{kl}\big]e_a^{*k},
\end{aligned}
\tag{13.11}
$$

where n_k and w^k are the momentum eigenvalues and the winding numbers in the direction k, respectively. One can show that the left and right moving contributions of the momentum in the compact dimensions, $p_{\mathrm{R},a}$ and $p_{\mathrm{L},b}$, have to satisfy the physicality condition

$$p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2 = 2w^k n_k \in 2\mathbb{Z}\,. \tag{13.12}$$

Thus one obtains an even and self-dual Lorentzian lattice $\Gamma^{(d,d)}$, where "Lorentzian" corresponds to the different signs of the metric in the two d-dimensional parts of the group.

The Lorentzian lattice $\Gamma^{(d,d)}$ should not be confused with the compactification lattice Λ^d although the same information is contained in both. The compactification lattice Λ^d describes the geometry, where each of the points in the Lorentzian lattice Γ additionally contains information about the momenta corresponding to particular winding numbers around each of the compact dimensions.

All even and self-dual Lorentzian lattices are related by $\mathrm{O}(d,d,\mathbb{R})$ rotations. Correspondingly, the group $\mathrm{O}(d,d,\mathbb{R})$ exhausts the full moduli space of toroidal backgrounds: all compactification configurations are related by those transformations.

However, not every element of the group $\mathrm{O}(d,d,\mathbb{R})$ leaves the spectrum of the theory and all correlators invariant. The subgroup of $\mathrm{O}(d,d,\mathbb{R})$ which leaves the spectrum invariant is $\mathrm{O}(d,d,\mathbb{Z})$. The complete physical group of symmetries of the considered d-dimensional toroidal compactifications is the group $\mathrm{O}(d,d,\mathbb{Z})$ combined with worldsheet parity $\sigma \to -\sigma$. Parity, which changes $B \to -B$, is not included in $\mathrm{O}(d,d,\mathbb{Z})$ because it corresponds to the interchange of p_{L} and p_{R}, which flips the sign of the Lorentzian norm $p_{\mathrm{R}}^2 - p_{\mathrm{L}}^2$.

13.2 Strong/Weak Coupling Duality: S-Duality

S-duality is the analogue of electric-magnetic duality. Its name originates from the fact that it relates the strong-coupling and weak-coupling limits of theories.

S-duality provides a way to access theories beyond perturbation theory. Consider a theory at small values of the coupling g_{s}: There are electrically charged elementary states which can be handled by perturbation theory. Likewise there are magnetically charged solitonic states, which are very massive and strongly coupled. For these heavy states, perturbation theory is not a good description.

Montonen and Olive proposed that for $g_{\mathrm{s}} \to \infty$ their roles might be reversed: they conjectured that one can reformulate a theory in terms of dual fields in such a way that the weakly coupled electrically charged states would turn into strongly coupled magnetic ones and vice versa.

In order to get a first impression of S-duality, let us consider the effective actions of the heterotic and the type I theory in ten dimensions. They read (in Einstein frame $G_{\mathrm{E},\mu\nu} = e^{-2\Phi} G_{\mu\nu}$):

$$S_{\mathrm{I}} \sim \int d^{10}x \sqrt{-\det G_{\mathrm{E}}}$$

$$\left[R_{\mathrm{E}} + 4(\partial\Phi)^2 - \tfrac{1}{12}e^{+4\Phi}H^2 + \tfrac{1}{4}e^{+2\Phi}F^2\right],$$

$$S_{\text{HET-O}} \sim \int d^{10}x\sqrt{-\det G_{\mathrm{E}}}$$

$$\left[R_{\mathrm{E}} + 4(\partial\Phi)^2 - \tfrac{1}{12}e^{-4\Phi}H^2 + \tfrac{1}{4}e^{-2\Phi}F^2\right], \qquad (13.13)$$

where the subscript E refers to the quantity in the Einstein frame and the field strengths F are associated to the gauge group SO(32) in either theory. Thus, the only difference between the two actions is the sign of the dilaton: a transformation $\Phi \to -\Phi$ relates the two actions. The dilaton is related to the string coupling g_{s} and thus this is an obvious hint towards a weak-strong duality. Considering the spectrum of perturbative and non-perturbative solutions, one can indeed show that S-duality relates the full theories.

Another example is the S-duality of type IIB string theory. The $\mathrm{SL}(2,\mathbb{R})$ invariance of the action is manifest, while S-duality mixes (perturbative) states from the NS-NS sector with (non-perturbative) states from the R-R sector and vice versa. The fundamental string in type IIB gets mapped to the D1-brane solution, while the solitonic five-brane is mapped onto the D5-brane.

S-Duality of the Heterotic String in $D = 4$. As this will become phenomenologically interesting, let us consider another example of a string compactification exhibiting S-duality. Compactifying the heterotic string theory on a 6-torus, one obtains $\mathcal{N} = 4$ supergravity coupled to $\mathcal{N} = 4$ Yang–Mills theory in four dimensions. The T-duality group is $\mathrm{O}(6, 22, \mathbb{Z})$. In addition there is a symmetry of the equations of motion: one can show that they are invariant under a $\mathrm{SL}(2,\mathbb{R})$ transformation. Quantum effects, however, break the $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{Z})$. It is not clear yet, whether the $\mathrm{SL}(2,\mathbb{Z})$ is a (non-perturbative) symmetry of the full action.

U-Duality: Combination of S- and T-Duality. Some string theories will exhibit T-duality and S-duality in their low-energy effective actions simultaneously. The prime example are the compactifications of the ten-dimensional IIB supergravity. Each of those compactifications exhibits at least one $\mathrm{SL}(2,\mathbb{Z})$ as well as $\mathrm{O}(d, d, \mathbb{Z})$ as subgroups of their complete symmetry groups. It is conjectured that the maximal integer subgroup of those two groups is a symmetry of the full string theory, in which case it is referred to as U-duality. The symmetry can be best explored on solutions to type IIA string theory; however, this is rather involved.

13.3 Strong Coupling Limits

Type IIA String Theory. In the strong coupling limit – which corresponds to a very large value of the dilaton – the type IIA string theory reveals its origin: the spectrum can be derived from an eleven-dimensional theory. The obvious candidate for this theory in the field theory limit is supergravity in eleven dimensions. Even more, one can show that type IIA perturbation theory is the expansion around the zero-radius limit of one dimension of $D = 11$ supergravity. Accordingly, the non-perturbative part of the spectrum of type IIA string theory contains all kinds of Kaluza–Klein modes originating from wrapping the eleven-dimensional solutions around the compact dimension. The resulting modes can be identified with type IIA D-branes. In fact, the whole spectrum of fundamental objects in type IIA string theory can be given an eleven-dimensional interpretation.

Heterotic $\mathbf{E_8 \times E_8}$ Theory. The strong coupling limit of the effective action of heterotic string theory with gauge group $E_8 \times E_8$ is again eleven-dimensional supergravity. In order to get from this supergravity back to the effective action of the heterotic string theory, however, one will have to compactify on an interval with length L or – equivalently – on a sector of a circle $S^1/\mathbb{Z}_2$. Correspondingly, the eleven-dimensional spacetime then consists of two nine-dimensional hyperplanes, separated by an interval of length L. Heterotic strings couple to those hyperplanes with gauge fields of E_8. If the effective actions of type IIA string theory as well as the heterotic $E_8 \times E_8$ theory are certain weak-coupling limits of eleven-dimensional supergravity, it is an obvious question, what the strong-coupling-limit of type IIA string theory and the heterotic theory corresponds to?

13.4 M-Theory

The concept of M-theory was suggested by Edward Witten in 1995 and initiated what is nowadays known as the "second superstring revolution". While Witten chose the letter "M" in M-theory for either "membrane", "magic" or "mysterious", everyone picks his or her own interpretation. M-theory is supposed to not be a string theory, but rather a non-perturbative theory of fundamental objects, whose low-energy limit is eleven-dimensional supergravity. The type I, type II and heterotic string theories can be thought of as different perturbative expansions at several points of the moduli space of M-theory.

A complete description of the dynamics of M-theory is not known. The best one can currently do is to formulate the low-energy dynamics of the theory in terms of eleven-dimensional supergravity interacting with two-dimensional and five-dimensional membranes. M-theory and its dynamics are still a field of active research.

14 String Theory and the Standard Model

In this chapter, some of the attempts of connecting string theory to the standard model will be discussed. In particular we will discuss, what can be learnt from string theory and what the current hopes and expectations are. In what sense does string theory answer the questions of quantum gravity?

14.1 The Real World

Standard Model and Gravity. Making connection to the real world first and foremost refers to reproducing the gauge group of the standard model along with the Higgs mechanism. The standard model, comprising three generations of quarks and leptons as well as the gauge bosons for the electromagnetic, the weak and the strong interactions and the Higgs particle is a non-abelian gauge theory with gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. So far, the standard model has passed all experimental tests performed. The standard model, however, appears to be rather arbitrary: there are around twenty constants which have to be adjusted to very high precision in order to reproduce the experimental findings. The origin of those constants remains unclear: it would be desirable, to have a theory not only predicting the general mechanisms underlying the standard model but as well the coupling strengths and masses of and between the constituents.

In terms of a gravitational theory, the state of the art is described very quickly: starting from Einstein gravity, the most successful model based on a Friedman–Robertson–Walker solution is the Λ-cold-dark-matter model. It is capable of describing the expansion of the universe in the way we observe it with the price of introducing *dark matter* and *dark energy*. So far the search for the dark matter has not been successful: no particle from the standard model does satisfy the bounds on mass (and thus coupling to gravity) as well as the other constraints originating in the observed interactions with visible matter. In addition, it is clear that the perturbative expansion of Einstein gravity is not sufficient to describe the physics in highly curved spaces such as they appear shortly after the big bang. In order to predict the processes in those regimes, a renormalisable theory for quantum gravity is unavoidable. Up to date, no renormalisable quantum field theory for gravity is known.

Supersymmetry. So far, supersymmetry has not been observed and is not part of the standard model. From the string theory perspective, however, supersymmetry is desirable in order to obtain a theory free of tachyons with stable D-branes, which in addition allows to maintain the ratio between the electroweak scale and the Planck mass. On the other hand, one can show that supersymmetric theories with $\mathcal{N} > 1$ do not allow for the chiral spectrum we observe. Balancing the two requirements leads to the goal of a four-dimensional theory with $\mathcal{N} = 1$ supersymmetry.

This theory should obviously contain the standard model. The minimal solution one can have is the so called *minimal supersymmetric standard model*. It contains all the particles of the standard model with the addition of a Higgs doublet. Each of the particles will be assigned a superpartner. All gauge interactions are fixed by the non-supersymmetric part of the standard model. All other (non-gauge) interactions are however not constrained, which leads in the smallest version to a model exhibiting even up to about a hundred constants to fix. Another problem is that the supersymmetric partners of the standard model particles will have the same masses as the original particles. If this was the case, one would have seen the superpartners already in collider experiments. So one will have to find a mechanism spreading the masses between superpartners.

Aiming at a theory which contains gravity as well as gauge interactions, the obvious strategy is to start from $\mathcal{N} = (1, 0)$ heterotic string theories in ten dimensions because these theories readily contain gauge groups which appear to be large enough to accommodate the standard model gauge group. The second ingredient allowing to shape the effective four-dimensional theory is a suitable compactification manifold: the structure of this manifold will – among many other things – determine the amount of supersymmetry present.

In practice, there are two ways to proceed: one can either construct a conformal field theory with suitable boundary conditions giving rise to a $\mathcal{N} = 1$ theory in four dimensions. This leads to the *orbifold compactifications*. The other way is to start with the effective supergravity action derived from the heterotic theory in ten dimension and then compactify on a suitable manifold: this will lead to *Calabi–Yau compactifications*.

14.2 Geometry of Toroidal Manifolds and Orbifolds

Compactifying a theory on a torus does not break supersymmetry. In order to produce the desired $\mathcal{N} = 1$ theory in four dimensions, one needs to think about a more sophisticated compactification manifold. One of the simplest generalisation of toric manifolds are *orbifolds*.

Orbifolds. An orbifold is a generalisation of a manifold: it is a quotient space of a Euclidean space by a finite group. Orbifolds will have *orbifold fixed points*, which are the points invariant under the identification. In the vicinity of those singular points, a quantum field (or string) theory defined on the orbifold will become singular itself, which will effectively reduce the number of states in the theory. In other words, respecting the symmetry group of the orbifold puts constraints on the theory defined on it. Simultaneously, at the singular points one can add new states to the theory: those go under the name of *twisted sectors* and they render a string theory completely smooth when combined with the regular parts.

In order to get used to the concept of an orbifold, let us start with an easy example. Take the real line $\mathbb{R}$, define a lattice

$$\Lambda = a\mathbb{Z} \tag{14.1}$$

and identify points via

$$x \simeq x + l \qquad \text{for all } x \in \mathbb{R}\,, l \in \Lambda\,. \tag{14.2}$$

The interval $0 \leq x < a$ is called the fundamental domain, while $\mathbb{R}$ is the covering space of the torus T. Now one defines a parity operation

$$Px = -x \qquad \text{for all } x \in \mathbb{R} \tag{14.3}$$

and identifies

$$x \simeq Px \qquad \text{for all } x \in \mathbb{R}\,. \tag{14.4}$$

Noticing that $P^2 = 1$, P is a realisation of the cyclic group $\mathbb{Z}_2$. The fixed points under the combined lattice and parity identifications are $x = 0$ and $x = 1/2$. The resulting space is the orbifold $T/\mathbb{Z}_2$.

In order to get from ten dimensions to four, one needs to choose a manifold of the form

$$O^6 = T^6/\mathrm{G}, \tag{14.5}$$

where, for simplicity, G is assumed to be a finite abelian point group. In the following we will work with the example $\mathrm{G} = \mathbb{Z}_3$. Conveniently, a point on this orbifold can be labelled by complex coordinates z_i with $i = 1, 2, 3$, which corresponds to splitting the torus into $T^6 = T_1^2 \times T_2^2 \times T_3^2$. An orbifold with this property is called factorisable.

In complete analogy to the physicality conditions for the periodicity conditions for the compactification of one dimension in the context of T-duality, the periodicity conditions on the orbifold read

$$z_i \sim z_i + 2\pi R_i \quad \text{and} \quad z_i \sim z_i + 2\pi R_i \rho_i, \tag{14.6}$$

where R_i are the radii of the tori and ρ_i are the corresponding complex structure moduli (which are the equivalent of the quantity τ in the discussion of the fundamental domain of the worldsheet).

In terms of the coordinates z_i one can define the orbifold action Θ, which reads

$$\Theta(z_i) = \exp(2\pi i \phi_i) z_i, \tag{14.7}$$

where the phases ϕ_i have to be integral multiples of $1/3$ in order to yield $\Theta^3 = 1$. The orbifold action needs to be compatible with the torus lattice, which in turn is defined by the complex structure moduli ρ_i. These conditions are pretty restrictive and allow to fix the phases ϕ uniquely to

$$\phi = \tfrac{1}{3}(1, 1, -2)\,. \tag{14.8}$$

Breaking of Supersymmetry. Let us now see, how the manifold $T^6/\mathbb{Z}_3$ described above actually will break the supersymmetry in the original theory in a way that the four-dimensional effective theory will have $\mathcal{N}=1$ supersymmetry. In ten dimensions, the supercharge Q is a 16-component spinor, which can be represented by a state $|s_{-1}, s_0, s_1, s_2, s_3\rangle$ with $s=\pm 1/2$ and $\prod_{i=-1}^{3} s_i = 1/2$. The orbifold action Θ acts as

$$\begin{aligned}\Theta|s_{-1}, s_0, s_1, s_2, s_3\rangle &= \exp\Big(2\pi i \sum_{i=1}^{3} \phi_i s_i\Big)|s_{-1}, s_0, s_1, s_2, s_3\rangle \\ &\stackrel{!}{=} |s_{-1}, s_0, s_1, s_2, s_3\rangle\,,\end{aligned} \tag{14.9}$$

and needs to leave the spinor invariant. For the values of ϕ_i fixed by the periodicity conditions on the orbifold, there is exactly one solution to the above equation ($\alpha=\pm 1$):

$$\big|s_{-1}, s_0, \tfrac{1}{2}\alpha, \tfrac{1}{2}\alpha, \tfrac{1}{2}\alpha\big\rangle. \tag{14.10}$$

Counting the number of independent spinor components of the above form reveals that the orbifold geometry indeed singles out four of the sixteen spinors in ten dimensions, which effectively leads to $\mathcal{N}=1$ supersymmetry in four dimensions.

One can construct a worldsheet conformal field theory with appropriate non-trivial boundary conditions, which will lead to this orbifold compactification in target space. From the worldsheet perspective this is still a free conformal field theory which can be solved exactly. This means in particular that one can derive the complete partition function and demand modular invariance, which will lead to additional constraints on the target space theory. The calculation is rather involved, but easy enough to allow for a complete classification of all possible orbifold conformal field theories originating from the orbifold $T^6/\mathbb{Z}_3$ starting from the heterotic string with gauge group SO(32).

The gauge groups of the resulting effective four-dimensional theories are very large and appear to be rather arbitrary. This unsatisfactory situation can be improved by introducing Wilson lines, which can be thought of as constant gauge fields in the string background. The introduction of these Wilson lines allows to equip the twisted sectors of different orbifold fixed points with different gauge groups. Correspondingly, the boundary conditions for the conformal field theory change. For the $T^6/\mathbb{Z}_3$-orbifold, one can add up to three Wilson lines.

There exists a $T^6/\mathbb{Z}_3$-model obtained from a heterotic string theory with gauge group $E_8 \times E_8$ with the field content of the minimal supersymmetric standard model. However, in addition to the desired particles, there are many vector-like exotic particles, which do not decouple completely and thus yield additional unwanted states.

14.3 Calabi–Yau Compactification of $D=10$ Supergravity

Let us just briefly comment on the second option pointed out above: one could as well start with the effective action of ten-dimensional supergravity coupled to ten-dimensional SYM theory and compactify this on a suitable manifold. As opposed to the orbifolds in the previous subsection, which are singular spaces, here one will use non-singular (or smooth) spaces.

Compactifying on a non-singular space, one still wants to obtain a $\mathcal{N} = 1$ supersymmetric model in four dimensions with three generations of leptons and quarks. The translation of those properties into constraints on the compactification manifold leads to a very special type of manifolds: the *Calabi–Yau* manifolds.

While numerous Calabi–Yau compactifications have been studied, the best known is the so-called *standard embedding*: it starts from the ten-dimensional effective action of the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string. After compactifying on a suitable Calabi–Yau manifold one will obtain a four-dimensional model with gauge group $\mathrm{E}_6 \times \mathrm{E}_8$. For some ranges of parameters, this model will yield the minimal supersymmetric standard model.

14.4 String Theory as a Phenomenological Model

The investigation of string compactifications is still an active field of research. However, finding a compactification yielding an effective model which is just the standard model has not been successful so far. If one allows for the minimal supersymmetric extension of the standard model, string theory can produce something similar, however, usually there are additional states which can not be consistently decoupled.

In terms of quantum gravity, string theory provides a structurally sound model for a theory of quantum gravity. Practically, however, experiments which would be capable of confirming the validity of the string theory predictions beyond the classical limit are insurmountable.

Overall, string theory appears to be a promising concept. However, the amount of engineering needed to relate string theories to realistic quantum field theories seems rather unnatural.

15 AdS/CFT Correspondence

The *AdS/CFT correspondence* is the (conjectured) exact duality between a string theory and a CFT in $2 \leq D < 10$, commonly a gauge theory:

- This is just remarkable!
- It provides a precise formulation of a string/gauge duality.
- It is a *holographic* duality in the sense that it relates theories in a different number of spacetime dimensions.
- There is a multitude of pairs of models related by the AdS/CFT correspondence.

15.1 Stack of D3-Branes

Let us motivate the main example of the AdS/CFT correspondence between IIB strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang–Mills (SYM) theory.

3-Brane Geometry. Consider the 3-brane solution of IIB supergravity

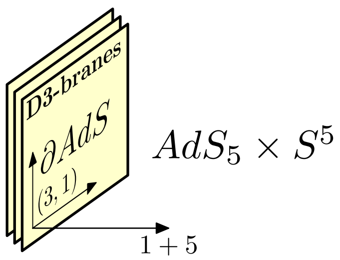
$$\begin{aligned} ds^2 &= h^{-1/2} dx^2 + h^{1/2} dy^2, \\ H_5 &= h^{-2} dh \wedge dx^{\wedge 4} + h^{-2} * (dh \wedge dx^{\wedge 4}). \end{aligned} \tag{15.1}$$

Here we have $3+1$ coordinates x along the brane and 6 extra coordinates y for the embedding space. The harmonic function $h(y) = 1 + \alpha N/|y|^4$ is spherically symmetric around the brane.

This geometry is a background for IIB string theory with a stack of N D3-branes. The low-energy brane dynamics is therefore described by $\mathrm{U}(N)$ $\mathcal{N} = 4$ SYM.

Now approach the brane at $y = 0$ or alternatively take the limit $N \to \infty$:

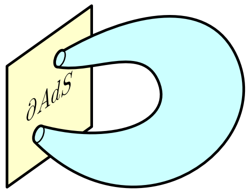
- The harmonic function limits to $h(y) = \alpha N/|y|^4$.
- The background becomes $AdS_5 \times S^5$ with 5-form flux.
- The sphere S^5 is given as submanifolds at constant $|y|$ and x. The $(4+1)$-dimensional AdS_5 spacetime is combined from the x coordinates and the distance $|y|$.



(15.2)

AdS/CFT Correspondence. Claims: (Maldacena)

- The 3-brane is at the boundary of the AdS_5 space.
- The low-energy string modes associated to the brane decouple from the rest.
- The boundary physics is described *exactly* by $\mathrm{U}(N)$ $\mathcal{N} = 4$ SYM.
- An open string on the boundary can probe the bulk $AdS_5 \times S^5$ string theory.



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- With a suitable dictionary, there is a precise matching of all observables in both models.
- There is a precise map of the coupling constants

$$(\kappa/R, g_\mathrm{s}) = (g_\mathrm{YM}^{1/2} N^{1/4}, g_\mathrm{YM}^2). \tag{15.4}$$

15.2 Anti-de Sitter Geometry

Let us briefly discuss the geometry of *anti-de Sitter* spacetime AdS_d:

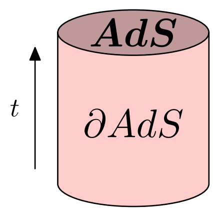
- It has constant scalar curvature but no tensorial curvature. It is a symmetric space, all points are equivalent.
- It is analogous to spheres and hyperbolic spaces as well as the *de Sitter space* according to the following table:

curvature	$+$	$-$
Euclidean	S	H
Minkowski	dS	AdS

(15.5)

- The isometry group is $\mathrm{SO}(d-1, 2)$. This is the same as the conformal group in $d-1$ dimensions.

Globally, it has the topology of a solid cylinder $\mathbb{R} \times D^{d-1}$.



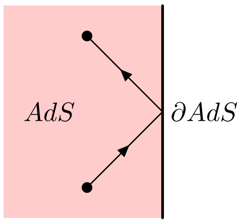
(15.6)

The boundary is the cylinder surface $\mathbb{R} \times S^{d-2}$.

There are some interesting facts about the geodesics in connection to the bulk and boundary:

- Time-like geodesics never reach the boundary.
- Space-like geodesics reach the boundary at infinite distance.

- Light-like geodesics reach the boundary in finite time. In this sense, the bulk and the boundary can interact via massless fields.



(15.7)

15.3 $\mathcal{N} = 4$ Super Yang–Mills

There is a unique maximally supersymmetric gauge theory in four dimensions: $\mathcal{N} = 4$ *super Yang–Mills* theory. It is the dimensional reduction of $D = 10$, $\mathcal{N} = 1$ SYM to $D = 4$. It contains the following fields:

- a gauge field,
- 4 adjoint chiral (or real) fermions,
- 6 adjoint real scalars.

It has a couple of remarkable properties:

- There is no running coupling, $\beta = 0$.
- It has exact 4D superconformal symmetry; 4D (S)CFT.
- ...

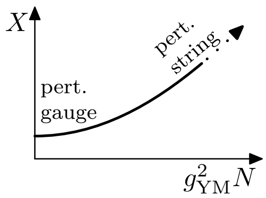
15.4 Tests

Evidently, we want to verify AdS/CFT correspondence. There are several useful predictions to be tested:

- The string spectrum matches with the spectrum of local operators.
- String and gauge correlation functions match.

A major problem in performing such tests is that the AdS/CFT correspondence is a *strong/weak duality*:

- Weakly coupled strings is strongly coupled gauge theory.
- Weakly coupled gauge theory is strongly coupled strings.

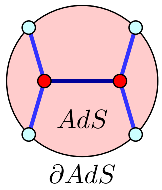


(15.8)

We have good means to compute observables at weak coupling, but there is no overlap between the weakly coupled regimes in both theories.

We can, however, test BPS quantities which are *protected* (independent of the coupling):

- Supergravity modes agree with BPS operators.
- Supergravity correlators match with BPS correlators.



(15.9)

What about other quantities?

- String and gauge theory appear to be *integrable* at large N.
- Integrability is a hidden symmetry which put strong constraints on the dynamics.
- One can compute observables efficiently even at finite coupling.
- This leads to a precise agreement in all tests that have been performed.

Other tests have been performed, for example the matching of circular Wilson loop expectation values and the area of a string worldsheet ending on this circle.