Quantum Information Theory
Solutions 11

HS 13
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Exercise 11.1 Information measures bonanza

Take a system A in state ρ. Non-conditional quantum min- and max-entropies are given by

$$H_{\min}(A)_\rho = -\log \max_{\lambda \in \mathrm{spec}(\rho)} \lambda, \qquad H_{\max}(A)_\rho = \log rank(\rho).$$

The mutual information measures correlations between two systems. For ρ_{AB}, we have

$$\begin{aligned} I(A:B)_\rho &= H(A)_\rho + H(B)_\rho - H(AB)_\rho \\ &= H(A)_\rho - H(A|B)_\rho. \end{aligned}$$

Show that if $\mathrm{spec}(\rho) \prec \mathrm{spec}(\tau)$, *then the entropy of ρ is larger than or equal to the entropy of τ, for the von Neumann, min- and max-entropies.* $\mathrm{spec}(\rho) \prec \mathrm{spec}(\tau)$ *means that* $\mathrm{spec}(\tau)$ *majorizes* $\mathrm{spec}(\rho)$. *See exercise 7.3 for more details.*

For simplicity, we define again $\vec{r} = \mathrm{spec}(\rho)$ and $\vec{t} = \mathrm{spec}(\tau)$, with the eivenvalues in decreasing order, and the sum-vectors from the previous exercise, $\vec{R} : R_k = \sum_{i=1}^k r_i$. We have

$$\begin{aligned} \vec{r} \prec \vec{t} \Rightarrow r_1 \leq t_1 \Leftrightarrow H_{\min}(A)_\rho \geq H_{\min}(A)_\tau \quad \checkmark \\ \vec{R}. \leq \vec{T} \wedge R_n = T_n = 1 \Rightarrow \left|\left\{1\text{'s in } \vec{R}\right\}\right| \leq \left|\left\{1\text{'s in } \vec{T}\right\}\right| \Leftrightarrow |\{0\text{'s in } \vec{r}\}| - 1 \leq \left|\left\{0\text{'s in } \vec{t}\right\}\right| - 1 \quad {}^* \\ \Leftrightarrow \mathrm{rk}(\rho) \geq \mathrm{rk}(\tau) \Leftrightarrow H_{\max}(A)_\rho \geq H_{\max}(A)_\tau \quad \checkmark \end{aligned}$$

* you can check the example from the last exercise to see that the number of ones in $\vec{R}$ equals the number of zeros in $\vec{r}$ minus one.

To prove that the same holds for the von Neumann entropy, we make use of its concavity, $H(A)_{\sum_k p_k \rho_k} \geq \sum_k p_k H(A)_{\rho_k}$. If $\mathrm{spec}(\rho) \prec \mathrm{spec}(\tau)$, we know that there exist $\{U_k, p_k\}_k$ such that $\rho = \sum_k p_k U_k \tau U_k^\dagger$. The von Neumann entropy for state ρ is

$$\begin{aligned} H(A)_\rho &= H(A)_{\sum_k p_k U_k \tau U_k^\dagger} \\ &\geq \sum_k p_k H(A)_{U_k \tau U_k^\dagger} \\ &= \sum_k p_k H(A)_\tau \quad \text{(entropy is invariant under unitaries)} \\ &= H(A)_\tau. \end{aligned}$$

Exercise 11.2 Davies' Theorem

Consider an arbitrary CQ state $\sigma^{XB} = \sum_x p_x |x\rangle\langle x|^X \otimes \rho_x^B$ and imagine making a measurement $\mathcal{M}$ having elements E_y on B. By the Holevo bound, $I(X{:}Y) \leq I(X{:}B) = S(\sum_x p_x \rho_x) - \sum_x p_x S(\rho_x)$. Define the accessible information $I_{\mathrm{acc}}(\sigma^{XB}) = \max_{\mathcal{M}} I(X:Y)$.

Show that the optimal measurement consists of rank-one elements and has no more than d^2 outcomes, where $d = \dim(B)$. Hint: the space of Hermitian operators on B is a vector space of size d^2.

Let us assume that the POVM set is mixed, such that $E_y = pE_y^{(1)} + (1-p)E_y^{(2)}$, where $\{E_y^{(1)}\}$ and $\{E_y^{(2)}\}$ are themselves valid POVM sets. Now let us measure the quantum part of the CQ state with this POVM:

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}}\, \sigma_{XB} = (\mathbb{1} \otimes E_y)\sigma_{XB} &= \sum_{xy} p_x |x\rangle\langle x|^X \otimes |y\rangle\langle y|^B \mathrm{tr}[\rho_x^B E_y] \\
&= \sum_{xy} p_x |x\rangle\langle x|^X \otimes |y\rangle\langle y|^B (p\, \mathrm{tr}[\rho_x^B E_y^{(1)}] + (1-p)\, \mathrm{tr}[\rho_x^B E_y^{(2)}]) \\
&= \sum_{xyz} p_x |x\rangle\langle x|^X \otimes |y\rangle\langle y|^B \otimes |z\rangle\langle z|^S \mathrm{tr}[\rho_x^B \otimes \sigma_z^S E_y^z]) \\
&= \mathrm{tr}_S[\mathcal{M}_{BS}\, \sigma_{XB} \otimes \sigma_S]
\end{aligned}$$

Here, we have introduced a fictitious system S such that $\mathrm{tr}[\sigma_S|0\rangle\langle 0|] = p$ and $\mathrm{tr}[\sigma_S|1\rangle\langle 1|] = 1-p$. It follows that for the mixed POVM sets finding the accessible informaton $I(X:Y)_{\mathcal{M}_B}$ after measuring the quantum system B is equivalent to the accessible information $I(X:YS)_{\mathcal{M}_{BS}}$ after measuring the quantum system B and the fictitious system S and tracing out over S. Since tracing out reduces the mutual information ($I(X{:}Y) \leq I(X{:}YZ)$), the most optimal case is when we do not have to trace the system S out, that is, when the POVM set is not mixed.

It remains to prove that we have a mixed POVM set. If we have a measurement $\mathcal{M}$ with $n > d^2$ elements E_y, there must exist a set of $q_y \neq 0$ such that $\sum_y q_y E_y = 0$ (E_y linearly dependent), which can be rescaled without loss of generality so that $-1 \leq q_y \leq 1$. Defining the measurements $\mathcal{M}_\pm = \{(1 \pm q_y)E_y\}$, which really are measurements since $\sum_y (1 \pm q_y)E_y = \mathbb{1}$, we have $\mathcal{M} = \frac{1}{2}(\mathcal{M}_+ + \mathcal{M}_-)$.

Exercise 11.3 Quantum Data Processing Inequality

Consider two CPTP maps $\$_1$ and $\$_2$ acting on system Q. Call the initial state of Q ρ^Q, the output of the first map $\rho^{Q'} = \$(\rho^Q)$ and the output of the second map $\rho^{Q''} = \$_2 \circ \$_1(\rho^Q)$. Purifying the initial state with a system R and using the Stinespring dilations of the CPTP maps, we can regard this transformation as taking the pure state Ψ^{RQ} to $\Psi^{RQ'E_1}$ and then to $\Psi^{RQ''E_1E_2}$, where E_1 (E_2) is the environment of the first (second) map, so that E_1E_2 is the environment of the concatenated map $\$_2 \circ \$_1$. Now define the coherent information $I(A\rangle B) = -S(A|B)$. Show that

$$S(Q) \geq I(R\rangle Q') \geq I(R\rangle Q'').$$

Hint: use (strong) subadditivity.

It has been shown in the lectures that for any bipartite pure state ϕ_{XY}, the entropy of the marginals is equal, i.e. $H(X) = H(Y)$. This result directly follows from Schmidt decomposition of the state. Note that this result can be used for any pure states, by imagining that it is a bipartite state. We will use, for example, that $H(B) = H(AC)$ for a pure state ψ_{ABC}.

The first inequality follows from subadditivity and the second from strong subadditivity. Observe that if C purifies AB, then $-S(A|B) = -S(AB) + S(B) = -S(C) + S(AC) = S(A|C)$, so $I(A\rangle B) = S(A|C)$. In the current context we have $I(R\rangle Q') = S(R|E_1) \leq S(R) = S(Q)$, where the last steps follow from the facts that system R is not involved in the transformation and system RQ is pure. For the second inequality we use strong subadditivity: $I(R\rangle Q'') = -S(R|E_1E_2) \geq -S(R|E_1) = I(R\rangle Q')$.