ETH

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Quantum Information Theory
Solutions 6

HS 13
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Exercise 6.1   Partial trace

Given a density matrix ρ_{AB} on the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_A = \mathrm{Tr}_B \rho_{AB}$,

a) Show that ρ_A is a valid density operator by proving it is:

1) *Hermitian: $\rho_A = \rho_A^\dagger$.*
   Remember that ρ_{AB} can always be written as

   $$\rho_{AB} = \sum_{i,j,k,l} c_{ij;kl}\, |i\rangle\langle k|_A \otimes |j\rangle\langle l|_B,$$

   where $c_{ij;kl} = c_{kl;ij}^\dagger$ is hermitian.
   The reduced density operator ρ_A is then given by

   $$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_{i,k}\sum_{m} c_{im;km}|i\rangle\langle k|_A$$

   as can easily be verified. Hermiticity of ρ_A follows from

   $$\rho_A^\dagger = \sum_{i,k}\sum_{m} c_{im;km}^\dagger \left(|i\rangle\langle k|_A\right)^\dagger = \sum_{i,k}\sum_{m} c_{km;im}|k\rangle\langle i|_A = \rho_A.$$

2) *Positive: $\rho_A \geq 0$.*
   Since $\rho_{AB} \geq 0$ is positive, its scalar product with any pure state is positive. Let $|\Psi_m\rangle_{AB} = |\psi\rangle_A \otimes |m\rangle_B$ be a state in $\mathcal{H}_A \otimes \mathcal{H}_B$ and $|\psi\rangle_A$ an arbitrary pure state in $\mathcal{H}_A$:

   $$0 \leq \sum_{m}\langle\Psi_m|\rho_{AB}|\Psi_m\rangle$$
   $$= \sum_{m}\langle\psi|_A \otimes \langle m|_B \rho_{AB}|\psi\rangle_A \otimes |m\rangle_B$$
   $$= \sum_{m}\sum_{i,j,k,l} c_{ij;kl}\langle\psi|i\rangle\langle k|\psi\rangle_A\langle m|j\rangle\langle l|m\rangle_B$$
   $$= \sum_{i,k}\sum_{m} c_{im;km}\langle\psi|i\rangle\langle k|\psi\rangle_A$$
   $$= \langle\psi|\rho_A|\psi\rangle$$

   Because this is true for any $|\psi\rangle$, it follows that ρ_A is positive.

3) *Normalised: $Tr(\rho_A) = 1$.*

   $$\mathrm{Tr}(\rho_A) = \sum_{i,j}\sum_{m,n} c_{im;km}\langle n|i\rangle\langle k|n\rangle$$
   $$= \sum_{m,n} c_{nm;nm} = \mathrm{Tr}(\rho_{AB}) = 1.$$

b) *Calculate the reduced density matrix of system A in the Bell state* $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

The reduced state is mixed, even though $|\Psi\rangle$ is pure:

$$\rho_{AB} = |\Psi\rangle\langle\Psi| = \frac{1}{2}\Big(|00\rangle\langle00| + |00\rangle\langle11| + |11\rangle\langle00| + |11\rangle\langle11|\Big)$$

$$\text{Tr}_B(\rho_{AB}) = \frac{1}{2}\Big(|0\rangle\langle0| + |1\rangle\langle1|\Big) = \frac{1}{2}\mathbb{1}_A.$$

c) *Consider a classical probability distribution* P_{XY} *with marginals* P_X *and* P_Y.

1) *Calculate the marginal distribution* P_X *for*

$$P_{XY}(x,y) = \begin{cases} 0.5 & \text{for } (x,y) = (0,0), \\ 0.5 & \text{for } (x,y) = (1,1), \\ 0 & \text{else,} \end{cases} \tag{1}$$

with alphabets $\mathcal{X}, \mathcal{Y} = \{0,1\}$.
Using $P_X(x) = \sum_y P_{XY}(x,y)$, we obtain

$$P_X(0) = 0.5, \quad P_X(1) = 0.5.$$

2) *How can we represent* P_{XY} *in form of a quantum state?*
A probability distribution $P_Z = \{P_Z(z)\}_z$ may be represented by a state

$$\rho_Z = \sum_z P_Z(z)|z\rangle\langle z|, \tag{2}$$

for a basis $\{|z\rangle\}_z$ of a Hilbert space $\mathcal{H}_Z$. In this case we can create a two-qubit system with composed Hilbert space $\mathcal{H}_X\mathcal{H}_Y$ in state

$$\rho_{XY}\frac{1}{2}\Big(|00\rangle\langle00| + |11\rangle\langle11|\Big).$$

3) *Calculate the partial trace of* P_{XY} *in its quantum representation.*
The reduces state of qubit X is

$$\rho_X = \frac{1}{2}\Big(|0\rangle\langle0| + |1\rangle\langle1|\Big).$$

Notice that the reduced states of this classical state and the Bell state are the same, the state of the global state is very different — in particular, the latter is a pure state that can be very useful in quantum communication and cryptography.

d) *Can you think of an experiment to distinguish the bipartite states of parts b) and c)?*

One could for instance measure the two states in the Bell basis,

$$|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad\qquad |\psi_2\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

$$|\psi_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \qquad\qquad |\psi_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

The Bell state we analised corresponds to the first state of this basis, $|\Psi\rangle = |\psi_1\rangle$, and a measurement in the Bell basis would always have the same outcome. For the classical state, however, $\rho_{XY} = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)$, so with probability $\frac{1}{2}$ a measurement in this basis will output $|\psi_2\rangle$, and we will know we had the classical state.

Exercise 6.2   State Distinguishability

One way to understand the cryptographic abilities of quantum mechanics is from the fact that non-orthogonal states cannot be perfectly distinguished.

a) *In the course of a quantum key distribution protocol, suppose that Alice randomly chooses one of the following two states and transmits it to Bob:*

$$|\phi_0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad or \qquad |\phi_1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + i|1\rangle). \tag{3}$$

Eve intercepts the qubit and performs a measurement to identify the state. The measurement consists of the orthogonal states $|\psi_0\rangle$ and $|\psi_1\rangle$, and Eve guesses the transmitted state was $|\phi_0\rangle$ when she obtains the outcome $|\psi_0\rangle$, and so forth. What is the probability that Eve correctly guesses the state, averaged over Alice's choice of the state for a given measurement? What is the optimal measurement Eve should make, and what is the resulting optimal guessing probability?

The the probability of correctly guessing, averaged over Alice's choice of the state is

$$p_{\text{guess}} = \tfrac{1}{2}(|\langle\psi_0||\phi_0\rangle|^2 + |\langle\psi_1||\phi_1\rangle|^2) \tag{4}$$

To optimize the choice of measurement, suppose $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$ for some $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. Then $|\psi_1\rangle = -\beta^*|0\rangle + \alpha^*|1\rangle$ is orthogonal as intended. Using this in (4) gives

$$p_{\text{guess}} = \frac{1}{2}\left(\left|\frac{\alpha^* + \beta^*}{\sqrt{2}}\right|^2 + \left|\frac{i\alpha - \beta}{\sqrt{2}}\right|^2\right) \tag{5}$$

$$= \tfrac{1}{2}(1 + 2\text{Re}\left[\left(\tfrac{1-i}{2}\right)\alpha\beta^*\right]). \tag{6}$$

If we express α and β as $\alpha = ae^{i\theta}$ and $\beta = be^{i\eta}$ for real a, b, θ, η, then we get

$$p_{\text{guess}} = \tfrac{1}{2}(1 + 2ab\text{Re}\left[\left(\tfrac{1-i}{2}\right)e^{i(\theta-\eta)}\right]). \tag{7}$$

To maximize, we ought to choose $a = b = \frac{1}{\sqrt{2}}$, and we may also set $\eta = 0$ since only the difference $\theta - \eta$ is relevant. Now we have

$$p_{\text{guess}} = \tfrac{1}{2}(1 + \text{Re}\left[\left(\tfrac{1-i}{2}\right)e^{i\theta}\right]) \tag{8}$$

$$= \tfrac{1}{2}(1 + \tfrac{1}{\sqrt{2}}\text{Re}\left[e^{-i\pi/4}e^{i\theta}\right]), \tag{9}$$

from which it is clear that the best thing to do is to set $\theta = \pi/4$ to get $p_{\text{guess}} = \tfrac{1}{2}(1 + \tfrac{1}{\sqrt{2}}) \approx$ 85.4%. The basis states making up the measurement are $|\psi_0\rangle = \tfrac{1}{\sqrt{2}}(e^{i\pi/4}|0\rangle + |1\rangle)$ and $|\psi_1\rangle = \tfrac{1}{\sqrt{2}}(-|0\rangle + e^{-i\pi/4}|1\rangle)$.

b) *Now suppose Alice randomly chooses between two states separated by an angle θ on the Bloch sphere. What is the measurement which optimizes the guessing probability? What is the resulting probability of correctly identifying the state?*

The point of this exercise is to show that thinking in terms of the Bloch sphere is a lot more intuitive than just taking a brute force approach as we did in the solution of the previous exercise. Let $\hat{n}_0$ and $\hat{n}_1$ be the Bloch vectors of the two states. Call $\hat{m}$ the Bloch vector associated with one of the two basis vectors of the measurement, specifically the

one which indicates that the state is $|\phi_0\rangle$ (the other is associated with $-\hat{m}$). The guessing probability takes the form

$$p_{\text{guess}} = \tfrac{1}{2}(|\langle\psi_0||\phi_0\rangle|^2 + |\langle\psi_1||\phi_1\rangle|^2) \tag{10}$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2}(1 + \hat{n}_0 \cdot \hat{m}) + \tfrac{1}{2}(1 - \hat{n}_1 \cdot \hat{m})\right) \tag{11}$$

$$= \tfrac{1}{4}\left(2 + \hat{m} \cdot (\hat{n}_0 - \hat{n}_1)\right) \tag{12}$$

The optimal $\hat{m}$ lies along $\hat{n}_0 - \hat{n}_1$ and has unit length, i.e.

$$\hat{m} = \frac{\hat{n}_0 - \hat{n}_1}{\sqrt{(\hat{n}_0 - \hat{n}_1) \cdot (\hat{n}_0 - \hat{n}_1)}} \tag{13}$$

$$= \frac{\hat{n}_0 - \hat{n}_1}{\sqrt{2 - 2\cos\theta}}. \tag{14}$$

Therefore,

$$p_{\text{guess}} = \tfrac{1}{4}\left(2 + \sqrt{2 - 2\cos\theta}\right) \tag{15}$$

$$= \tfrac{1}{2}\left(1 + \sqrt{\frac{1 - \cos\theta}{2}}\right) \tag{16}$$

$$= \tfrac{1}{2}\left(1 + \sin\tfrac{\theta}{2}\right). \tag{17}$$

Finally, we should check that this gives sensible results. When $\theta = 0$, $p_{\text{guess}} = \tfrac{1}{2}$, as it should. On the other hand, the states $|\phi_k\rangle$ are orthogonal for $\theta = \pi$, and indeed $p_{\text{guess}} = 1$ in this case. In the previous exercise we investigated the case $\theta = \tfrac{\pi}{2}$ and here we immediately find $p_{\text{guess}} = \tfrac{1}{2}(1 + \tfrac{1}{\sqrt{2}})$, as before.

Exercise 6.3   Fidelity

a) *Given a qubit prepared in a completely unknown state $|\psi\rangle$, what is the* fidelity *F of a random guess $|\phi\rangle$, where $F(|\phi\rangle, |\psi\rangle) = |\langle\phi||\psi\rangle|^2$? The fidelity can be thought of as the probability that an input state (the guess) $|\phi\rangle$ passes the "ψ" test, which is the measurement in the basis $|\psi\rangle$, $|\psi^\perp\rangle$.*

First let's just try to guess the result. The unknown state $|\psi\rangle$ is somewhere on the Bloch sphere, and we might as well orient the sphere so that this direction is the $\hat{z}$ direction. The fidelity of $|\psi\rangle$ with any other state $|\phi\rangle$ is given by

$$|\langle\psi||\phi\rangle|^2 = \text{Tr}[P_\psi P_\phi] = \tfrac{1}{2}(1 + \cos\theta), \tag{18}$$

where θ is the angle between the two states on the Bloch sphere. Any state in the $\hat{x}$-$\hat{y}$ plane has a fidelity of $\tfrac{1}{2}$, and since a random state is as likely to lie in the upper hemisphere as in the lower, i.e. $\theta = \tfrac{\pi}{2} + \alpha$ and $\theta = \tfrac{\pi}{2} - \alpha$ are equally-likely, the average fidelity ought to be $\tfrac{1}{2}$. A simple integration confirms this guess:

$$\langle F\rangle = \frac{1}{4\pi}\int_0^{2\pi}\phi\int_0^\pi\theta\sin\theta\,\frac{1}{2}(1 + \cos\theta) = \frac{1}{4}\int_0^\pi\theta\sin\theta = \frac{1}{2}. \tag{19}$$

b) *In order to improve the guess, we might make a measurement of the qubit, say along the $\hat{z}$ axis. Given the result $k \in \{0, 1\}$, our guess is then the state $|k\rangle$. What is the average fidelity of the guess after the measurement, i.e. the probability of passing the "ψ" test?*

Given the outcome $|k\rangle$, the fidelity is $F_k = |\langle k||\psi\rangle|^2$ and this occurs with probability $p_k = |\langle k||\psi\rangle|^2$, so averaging over the measurement outcome gives $F = \sum_k p_k F_k = \sum_k |\langle k||\psi\rangle|^4$. Now we average over $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$:

$$\langle F\rangle = \frac{1}{4\pi}\int_0^{2\pi}\phi\int_0^{\pi}\theta\sin\theta\left(\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2}\right) = \frac{2}{3}. \tag{20}$$

Thus making the measurement increases the fidelity of the guess.