Quantum Information Theory
Series 12

HS 13
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Exercise 12.1 Entanglement and Teleportation

Quantum teleportation from Alice A to Bob B can be described by a linear map $\mathcal{E}$ from operators on $\mathcal{H}_A$ to operators on $\mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_B$ are copies. In the lecture we showed that $\mathcal{E}[|\psi\rangle\langle\psi|] = |\psi\rangle\langle\psi|$ for all pure states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

a) Look at the state of the system after Alice's measurement but before she communicates her results to Bob. At this point, we know that Bob's state is in one of four possible states that are related to $|\psi\rangle$. Show that we still cannot extract any information on $|\psi\rangle$ out of Bobs state by calculating it as a probabilistic mixture of the four possible states. What is the physical relevance of this observation?

b) Show that the pure states span the space of Hermitian matrices.

c) Show that $\mathcal{E}[\rho] = \rho$, for any mixed state ρ. Furthermore, show that $(\mathcal{E} \otimes \mathbb{1}_R)[|\Psi\rangle\langle\Psi|] = |\Psi\rangle\langle\Psi|$, for any $|\Psi\rangle$ on $\mathcal{H}_A \otimes \mathcal{H}_R$. This implies that quantum teleportation preserves entanglement!

Exercise 12.2 Resource inequalities: teleportation and classical communication

We saw a protocol, teleportation, to transmit one qubit using two bits of classical computation and one ebit, $[qq] + 2[c \to c] \geq [q \to q]$. Now suppose that Alice and Bob share unlimited entanglement: they can use up as many ebits as they want. Can Alice send n qubits to Bob using less than $2n$ bits of classical communication? In other words, we want to know if the following is possible:

$$m[c \to c] + \infty[qq] \geq n[q \to q] + \infty[qq], \qquad m < 2n$$

Prove that this is not the case. **Hint:** use superdense coding.

Exercise 12.3 A sufficient entanglement criterion

In general it is very hard to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that correctly identifies all entangled states in low dimensions.
Recall that we say that a bipartite state ρ_{AB} is separable (not entangled) if

$$\rho = \sum_k p_k \; \sigma_k \otimes \tau_k, \quad \forall k : p_k \geq 0, \sigma_k \in \mathcal{S}_=(\mathcal{H}_A), \tau_k \in \mathcal{S}_=(\mathcal{H}_B), \quad \sum_k p_k = 1.$$

a) Let $\Lambda_A : \mathrm{End}(\mathcal{H}_A) \mapsto \mathrm{End}(\mathcal{H}_A)$ be a positive map. Show that $\Lambda_A \otimes \mathcal{I}_B$ maps separable states to positive operators.

This means that if we apply $\Lambda_A \otimes \mathcal{I}_B$ to a bipartite state ρ_{AB} and obtain a non-positive operator, we know that ρ_{AB} is entangled. In other words, this is a sufficient criterion for entanglement.

b) Apply the partial transpose, $\mathcal{T}_A \otimes \mathcal{I}_B$, to the ϵ-noisy Bell state

$$\rho^\epsilon_{AB} = (1-\epsilon)\; |\psi^-\rangle\langle\psi^-| + \epsilon\; \frac{\mathbb{1}_4}{4}, \quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad \epsilon \in [0,1].$$

For what values of ϵ can we be sure that ρ^ϵ is entangled?

Note: As shown earlier, the transpose map is defined by $\mathcal{T}\left(\sum_{ij} a_{ij} |i\rangle\langle j|\right) = \sum_{ij} a_{ji}\; |i\rangle\langle j|$,

Remark: Indeed, it can be shown that the PPT criterion (positive partial transpose) is necessary and sufficient for systems of dimension 2×2 and 2×3.