Quantum Information Theory
Series 10

HS 13
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Exercise 10.1 Teleportation Redux

(a) Show that for the entangled state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and any unitary operator U,

$$\left(U_A \otimes \overline{U}_B\right)|\Phi\rangle_{AB} = |\Phi\rangle_{AB},$$

where $^-$ denotes complex conjugation in the $|0\rangle, |1\rangle$ basis.

(b) Show that for any state $|\psi\rangle$

$${}_A\langle\psi|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}|\psi^*\rangle_B.$$

(c) Use the results of (a) and (b) to give a derivation of the teleportation protocol without resorting to components.

(d) What happens if Alice and Bob use the state $(\mathbb{1}_A \otimes U_B)|\Phi\rangle_{AB}$ for teleportation? Or if Alice measures in the basis $\overline{U}_{A'}|\Phi_j\rangle_{A'A}$?

(e) Instead of a single system state $|\psi\rangle_{A'}$, Alice has a bipartite state $|\psi\rangle_{A_1A_2}$. What happens if she performs the teleportation protocol on system A_2?

Exercise 10.2 Remote Copy

Alice and Bob would like to create the state $|\Psi\rangle_{AB} = a|00\rangle_{AB} + b|11\rangle_{AB}$ from Alice's state $|\psi\rangle_A = a|0\rangle_A + b|1\rangle_A$, a "copy" in the quantum-mechanical sense. Additionally, they share the canonical entangled state $|\Phi\rangle$. Can they create the desired state by performing only local operations (measurements and unitary operators), provided Alice can only send *one* bit of classical information to Bob?

Exercise 10.3 Quantum mutual information

Consider a composed system $A \otimes B \otimes C$ with a shared state ρ_{ABC}.
In a first step we ignore system C and consider only $A \otimes B$ (and the reduced state $\rho_{AB} = \mathrm{Tr}_C(\rho_{ABC})$). One way of quantifying the correlations between A and B is to use the *mutual information* between them, defined as

$$\begin{aligned} I(A:B) &= H(A) + H(B) - H(AB) &(1)\\ &= H(A) - H(A|B). &(2)\end{aligned}$$

If we have access to C, we can define a conditional version of the mutual information between A and B as

$$\begin{aligned} I(A:B|C) &= H(A|C) + H(B|C) - H(AB|C) &(3)\\ &= H(A|C) - H(A|BC). &(4)\end{aligned}$$

(a) Assume a system formed by two qubits A and B that share a state ρ_{AB}. Consider bases $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ for the subsystems of each qubit.

1. Check that the mutual information of the fully entangled state , $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, is maximal.

2. See that for classically correlated states, $\rho_{AB} = p|0\rangle\langle 0|_A \otimes \sigma_B^0 + (1-p)|1\rangle\langle 1|_A \otimes \sigma_B^1$ (where $0 \leq p \leq 1$), the mutual information cannot be greater than one.

(b) Consider the so-called *cat state* shared by four qubits, $A \otimes B \otimes C \otimes D$, that is defined as

$$|☺\rangle = \frac{1}{\sqrt{2}} \left(|0000\rangle + |1111\rangle \right). \tag{5}$$

Check how the mutual information between qubits A and B changes with the knowledge of the remaining qubits, namely:

1. $I(A:B) = 1$.
2. $I(A:B|C) = 0$.
3. $I(A:B|CD) = 1$.