Quantum Information Theory
Series 9

HS 13
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Exercise 9.1 The Choi Isomorphism

The Choi Isomorphism can be used to determine whether a given mapping is a CPM. Consider the family of mappings between operators on two-dimensional Hilbert spaces

$$\mathcal{E}_\alpha : \rho \;\mapsto\; (1-\alpha)\,\frac{\mathbf{1}_2}{2} + \alpha\,\big(\frac{\mathbf{1}_2}{2} + X\rho Z + Z\rho X\big), \qquad 0 \le \alpha \le 1. \tag{1}$$

a) Use the Bloch representation to determine for what range of α these mappings are positive. What happens to the Bloch sphere?

b) Calculate the analogs of these mappings in state space by applying the mappings to the fully entangled state as follows:

$$\sigma_\alpha = (\mathcal{E}_\alpha \otimes \mathcal{I})\big[|\Psi\rangle\langle\Psi|\big], \qquad |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2}$$

For what range of α is the mapping a CPM?

c) Find an operator-sum representation of $\mathcal{E}_\alpha$ for $\alpha = 1/4$.

Exercise 9.2 Uncertainty relations

In Section 6.2 in the script, we consider the measurements (cf. (6.24) and (6.25))

$$\tilde{\Gamma}_x = W^*(\tilde{P}_x \otimes \mathrm{id})W \quad \text{and} \tag{3}$$

$$\Gamma_z = W^*(\mathrm{id} \otimes P_z)W, \tag{4}$$

with W, $\tilde{P}_x$ and P_z as defined in the script. Show that they can be written as (6.26) and (6.27), i.e.,

$$\Gamma_z = \frac{1}{2}\left(\mathrm{id} + (-1)^z \cos 2\theta \sigma_z\right) \tag{5}$$

$$\tilde{\Gamma}_x = \frac{1}{2}\left(\mathrm{id} + (-1)^x \sin 2\theta \sigma_x\right). \tag{6}$$

Exercise 9.3 "All-or-Nothing" Violation of Local Realism

Consider the three qubit state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right)_{123}$, the Greenberger-Horne-Zeilinger state.

a) Show that $|\mathrm{GHZ}\rangle$ is a simultaneous eigenstate of $X_1Y_2Y_3$, $Y_1X_2Y_3$, and $Y_1Y_2X_3$ with eigenvalue $+1$, where X and Y are the corresponding Pauli operators.

b) Use the results of part (a) to argue by Einstein locality that each qubit has well-defined values of X and Y. For qubit j, denote these values by x_j and y_j. We say that these values are *elements of reality*. What would local realism, i.e. the assumption of realistic values that are undisturbed by measurements on other qubits, predict for the product of the outcomes of measurements of X on each qubit?

c) What does quantum mechanics predict for the product of the outcomes of measurements of X on each qubit?