Exercise 1. ***Evolution operator***

The interaction picture field operator $\phi_I(x)$ is related to the full Heisenberg field operator $\phi_H(x)$ by $\phi_H(t,\vec{x}) = U^\dagger(t,t_0)\phi_I(t,\vec{x})U(t,t_0)$ with $U(t,t_0) = e^{iH_0\,(t-t_0)}e^{-iH\,(t-t_0)}$.

(a) Show that $U(t,t_0)$ satisifies the differential equation

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0) \tag{1}$$

with the initial condition $U(t_0,t_0) = 1$. Here $H_I(t)$ is the interaction Hamiltonian in the interaction picture.

(b) Show that the unique solution to this equation with the same initial condition can be written as

$$U(t,t_0) = \mathrm{T}\exp\left(-i\int_{t_0}^{t} dt' H_I(t')\right)$$

(c) Show that the operator satisfies $U^\dagger(t,t_0) = U(t_0,t)$ and $U(t_1,t_0)U(t_0,t_2) = U(t_1,t_2)$.

Exercise 2. ***Wick's theorem***

Wick's theorem relates the time-ordered product of fields $\phi(x) \equiv \phi_I(x)$ to the normal-ordered product plus all possible contractions

$$T\big(\phi(x_1)\dots\phi(x_m)\big) = :\phi(x_1)\dots\phi(x_m) + \text{all contractions}:$$

Prove this theorem by induction. What does change in the case of fermionic operators?

Exercise 3. ϕ^3 ***theory***

Consider a scalar theory with an interaction (notation: $\phi(x) \equiv \phi_I(x)$)

$$\mathcal{H}_I = \frac{\lambda_0}{3!}\,\phi^3$$

Find all contributions to

$$\langle 0|T\big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)e^{-i\int d^4x\,\mathcal{H}_I}\big)|0\rangle$$

up to second order in λ_0. Find the symmetry factor of all (connected, disconnected and vacuum-to-vacuum) diagrams. Show that for the Green function

$$G(x_1,x_2,x_3,x_4) = \frac{\langle 0|T\big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,e^{-i\int d^4x\,\mathcal{H}_I}\big)|0\rangle}{\langle 0|T\big(e^{-i\int d^4x\,\mathcal{H}_I}\big)|0\rangle}$$

the vacuum-to-vacuum diagrams are eliminated.