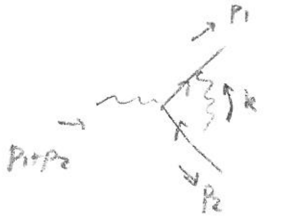
7. Loops & Renormalization

7.1. UV-singularities

Loop integrals involve integrations over internal momenta.
These integrals often do <u>not</u> converge!

e.g.



$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\rho (\not{p}_1 - \not{k} + m_0)\, \gamma^\mu\, (-\not{p}_2 - \not{k} + m_0)\, \gamma_\rho}{k^2 \left[(p_1-k)^2 - m_0^2\right] \left[(p_2+k)^2 - m_0^2\right]}$$

$$\overset{k\to\infty}{\sim} \int \frac{d^4k\; k^2}{k^6} \sim \int_0^{\Lambda\to\infty} \frac{dk}{k} \sim \lim_{\Lambda\to\infty} \ln\Lambda$$

There is a singularity associated with $k \to \infty$ (here logarithmic sing)
$\to$ UV singularity (ultra violet)

Note: there is another class of sing., associated with $k \to 0$
these are IR singularities (infra red)
at the moment we're not concerned with IR sing.

Which diagrams in QED are divergent?

Start by defining <u>superficial degree of divergence</u> (SDOD)

SDOD $\equiv$ # k in numerator − # k in denominator

Consider L-loop diagram with V vertices, I_f internal fermion lines, I_b internal photon lines, E_f external fermions and E_b external photons

$$\text{SDOD} = \underbrace{4L}_{\int d^4k} - \underbrace{I_f}_{\sim \frac{k}{k^2}} - \underbrace{2I_b}_{\sim \frac{1}{k^2}}$$

but $L = \underbrace{I_b + I_f}_{\#\text{ integrations } \int d^4q} - \underbrace{V}_{\#\ \delta^4(\sum q_i)} + \underbrace{1}_{\text{overall 4-mom. conservation}}$ (count internal momenta)

and  only vertex

$$V = 2I_b + E_b \rightarrow I_b = \frac{V-E_b}{2}$$
$$2V = 2I_f + E_f \rightarrow I_f = V - \frac{E_f}{2}$$

$$\text{SDOD} = \underbrace{4I_b + 4I_f - 4V + 4}_{4L} - I_f - 2I_b = \underbrace{4 - E_b - \tfrac{3}{2}E_f}_{\text{depends only on \# external lines !!!}}$$

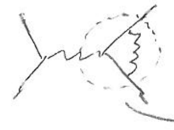
in 4-dim. SDOD does <u>not</u> depend on internal structure of diagram

SDOD ≥ 0 diagram superficially(!) divergent

SDOD = 2 quadratic div
SDOD = 1 linear div.
SDOD = 0 logarithmic div.

SDOD < 0 diagram superficially (!) convergent

Note !! SDOD is <u>NOT</u> actual degree of divergence
(still a very useful concept)

<u>example</u>  $\text{SDOD} = 4 - 0 - \frac{3}{2}\cdot 4 = -2$

but diagram divergent!

diagrams with SDOD < 0 can have divergent subdiagrams

<u>example</u>  $\text{SDOD} = 4 - 3 - \frac{3}{2}\cdot 0 = 1$

but diagram is finite, actually $= 0$.

Symmetries can make diagrams "more convergent" than expected

Furry's theorem  $= 0$ for n odd

$$\langle 0 | \underset{\substack{\wedge \\ C^2=1}}{A_{\mu_1}(x_1)} \cdots \underset{\substack{\wedge \\ C^2=1}}{A_{\mu_n}(x_n)} \cdot e^{-i \int d^4x\, H_I(x)} |0\rangle$$

use $C|0\rangle = |0\rangle$

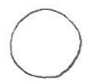
and $C H_I(x) C = H_I$ → QED is invariant under C

$$= \langle 0 | (C A_{\mu_1}(x_1) C) \cdots (C A_{\mu_n}(x_n) C) \cdot e^{-i \int d^4x\, H_I(x)} |0\rangle$$

$$= (-1)^n \langle 0 | A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)\, e^{-i \int d^4x\, H_I(x)} |0\rangle$$

$\Rightarrow$ for n odd $\langle 0 | A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)\, e^{-i \int d^4x\, H_I(x)} |0\rangle = 0$

Superficially divergent QED amplitudes



$E_b = 0$, $E_f = 0$
SDOD $= 4$

but does not contribute to S-matrix ("only" shift in vacuum energy)

$E_b = 1$ $E_f = 0$
SDOD $= 3$

but 0 due to Furry's theorem

$E_b = 0$ $E_f = 2$
SDOD $= 1$

but: $\sim A_0 + A_1 \not{p} + A_2 p^2 + \dots$

A_0: lin div. $\sim m$ → will have log div only (→ later); $A_1 \not{p}$: log. div; $A_2 p^2 + \dots$: finite

$E_b = 3$ $E_f = 0$
SDOD $= 1$

but 0, due to Furry

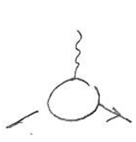
$E_b = 4$ $E_f = 0$
SDOD $= 0$

gauge invariance "improves" convergence
actually finite!

$E_b = 2$ $E_f = 0$
SDOD $= 2$

logarithmic divergence

$\sim (g^{\mu\nu} p^2 - p^\mu p^\nu)$ to ensure $p_\mu M^\mu = 0$



$E_b = 1$ $E_f = 2$
SDOD $= 0$

logarithmic divergence

the UV-singularities in QED in 4 dimensions are "organised" into 4 structures; all are log-div.

 $\sim A_0 + A_1 \not p$

$\hookrightarrow$ wave function renormalization of ψ $\to Z_2$

$\hookrightarrow \sim m$ $\to$ mass renormalization $\to Z_m$

 $\sim (g^{\mu\nu} p^2 - p^\mu p^\nu) A_0$

$\hookrightarrow$ wave function renormalization of A $\to Z_3$

 Vertex renormalization $\to Z_1$

$\hookrightarrow$ coupling renormalization

Consider QED in $D \neq 4$ dimension

$$\text{SDOD} = D \cdot L - I_f - 2 I_b = D + \underbrace{\frac{D-4}{2} \cdot V}_{} - \frac{D-2}{2} E_b - \frac{D-1}{2} E_f$$

related to dimension of coupling ! $[e] = \frac{4-D}{2}$

$[S] = [\int d^Dx\, \mathcal{L}] = 0 \quad (\hbar = 1) \quad \Rightarrow [\mathcal{L}] = D$

$$\left.\begin{array}{lll} [(\partial_\mu A^\mu)^2] = D \;\&\; [\partial_\mu] = 1 & \Rightarrow & [A^\mu] = \frac{D-2}{2} \\ [m \bar\psi \psi] = D \;\&\; [m] = 1 & \Rightarrow & [\psi] = \frac{D-1}{2} \end{array}\right\} [e \bar\psi A \psi] = D \Rightarrow [e] = \frac{4-D}{2}$$

for <u>$D>4$, coupling negative mass dimension</u>: diagrams become ever more divergent (with increasing nr of vertices V) $\to$ <u>non renormalizable</u>

(negative mass dim of coupling $\to$ higher mass dim of int. mon.)

<u>for $D<4$,</u> (coupling positive mass dim) only a finite nr. of <u>diagrams</u> is (superficially) divergent $\to$ <u>super renormalizable</u>

(increase V, at some point all diag are superficially finite)

<u>for $D=4$</u> (coupling dim less) infinite nr. of diagrams is divergent but, a finite nr. of amplitudes is superficially div. $\to$ <u>renormalizable</u> only

7.2 Renormalization

computing some amplitude (say in QED)

$$\mathcal{A}_i \sim \int d^4k\, F_i(k, m_0, e_0) \sim \int_0^\infty dk\, k^3 F_i(k, m_0, e_0) \sim \text{"} = \infty \text{"}$$

interpretation of "∞": QED is only valid up to some scale Λ

i.e. view QED as effective theory (e.g. gravity not included, so $\Lambda \lesssim M_{Planck}$)

$$\Rightarrow \mathcal{A}_i \sim \underbrace{\int_0^\Lambda dk\, k^3 F_i(k)}_{\text{"ok"}} + \underbrace{\int_\Lambda^\infty dk\, k^3 F_i(k)}_{\text{"wrong" not described by QED}} \sim \mathcal{A}_i^{eff}(\Lambda) + \mathcal{A}_i^{div}(\Lambda)$$

This does not mean QED is not predictable!

from section 7.1: we have n "infinite terms" cannot be predicted within QED but can be expressed in terms of measured quantities (e.g. mass of electron, coupling electron - photon etc.)

generally: finite number of infinite terms
QED: n=4

$\Rightarrow$ fix n renormalization conditions $j = 1, n$

$\Rightarrow$ define renormalized parameters (fields, couplings, masses)

$$\mathcal{A}_j = \mathcal{A}_j^{eff} + \mathcal{A}_j^{div} \equiv \text{input}_j$$

$\rightarrow$ determine ("measure") $\mathcal{A}_j^{div}$

$\Rightarrow$ compute any other observable $\mathcal{A}_k$ in terms of renormalized quantities: they will all be finite!

e.g. $\underbrace{m_0}_{\text{bare mass}} = \underbrace{m}_{\text{renormalized mass}} + \underbrace{m_{div}}_{\text{divergent part}}$

$\mathcal{A}_k(m_0 \ldots)$ contains divergences, but $\mathcal{A}_k(m, \ldots)$ has no divergences!

Remarks

- with finite number of inputs, can compute any other quantity in terms of these input parameters
- this works at all orders in perturbative expansion (in renormalizable theory)
- Split of $\int d^4k$ into finite + divergent is usually not done as $\int_0^{\Lambda} dk + \int_{\Lambda}^{\infty} dk$, but other techniques (dimensional regularization). Still, this will introduce a mass scale
- choice of (measured) input (i.e. renormalization conditions) not unique → choice of renormalization scheme
- results computed in different ren. schemes are equivalent within the approximation made (i.e. if we compute up to $\mathcal{O}(\alpha^n) = \mathcal{O}(e^{2n})$ difference is $\mathcal{O}(\alpha^{n+1})$)
- changes in Λ are governed by renormalization group
- relations between observables and bare quantities contain divergences. This is not a problem, as bare quantities are not observables
- ⌈ even non-renormalizable theories can be predictive i.e. give relations between observables but here we deal only with renormalizable theories ⌋

Renormalization of ϕ^4

Illustrate main ideas in ϕ^4 (SDoD = 4 − E_b : exercise)

define renormalized field ϕ_R, mass $m_R \equiv m$ and coupling $\lambda_R = \lambda$

through $\quad \phi = \sqrt{Z}\ \phi_R \qquad m_0 = Z_m \cdot m_{(R)} \qquad \lambda_0 = Z_1 Z_\phi^{-2} \lambda_{(R)}$

Define $m_{(R)}$ and $\lambda_{(R)}$ to be observables!

→ compute them in terms of bare quantities (→ divergences)

→ absorb divergences into renormalization factors Z

→ multiplicative renormalization

equivalent : rewrite Lagrangian (→ renormalized pert. theory)

$$\rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{m^2}{2}\phi_R - \frac{\lambda}{4!}\phi_R^4$$

$$+ \frac{1}{2}\underbrace{(Z-1)}_{\delta_Z}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}\underbrace{(Z_m^2 Z - 1)m^2}_{\delta_m}\phi_R^2 - \underbrace{\frac{\lambda}{4!}(Z_1 - 1)}_{\frac{\delta_\lambda}{4!}}\phi_R^4$$

counter terms → additional Feynman rules

↳ $-\otimes- \; = \; i(p^2\delta_Z - \delta_m) \qquad \times\!\!\!\!\otimes \;\; -i\delta_\lambda$

"modified" Feynman rules $\quad$ —— $= \dfrac{i}{p^2 - m^2 + i0^+} \qquad \times \;\; -i\lambda$

renormalized quantities!

fix counterterms (@ each order in pert. th.) such that the following renormalization conditions are satisfied

$$-\!\!\bigcirc\!\!- \; = \frac{i}{p^2 - m^2} + \text{terms regular @ } p^2 = m^2$$

$$\times\!\!\!\!\bigcirc \; = -i\lambda \quad \text{(e.g. at } s = 4m^2,\ t = u = 0)$$

with m^2 & λ "measured" physical quantities

def. $-(\text{1PI})- \; = -iM^2(p^2)$, then $-\!\!\bigcirc\!\!- \; = \dfrac{i}{p^2 - m^2 - M^2(p^2)}$ $\quad$ %. →

ren. conditions: $M^2(p^2 = m^2) = 0$ and $\dfrac{d}{dp^2}M(p^2)\Big|_{p^2 = m^2} = 0$ ← residue = 1

position of pole

QED:

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi_0} (\partial_\mu A^\mu)^2 - m_0 \bar{\psi}\psi - ie_0 \bar{\psi} \not{A} \psi - i \bar{\psi} \not{\partial} \psi$$

define renormalized fields, masses and couplings

$$\psi = \sqrt{Z_2}\, \psi_R \qquad A^\mu = \sqrt{Z_3}\, A^\mu_R \qquad (\xi_0 = Z_\xi \cdot \xi)$$

$$m_0 = Z_m\, m_{(R)} \qquad e_0 = \frac{Z_1}{Z_2 \sqrt{Z_3}}\, e_{(R)}$$

Z_2, Z_3 (wave function renormalization) Z_1, Z_m (and Z_ξ) absorb all UV singularities. Finite part of Z_i fixed by renormalization scheme

rewrite $\mathcal{L}_{QED}$

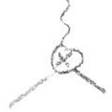
$$\mathcal{L}_{QED} = -\frac{1}{4} F^R_{\mu\nu} F^{\mu\nu}_R + \bar{\psi}_R (i\not{\partial} - m) \psi_R - e \bar{\psi}_R \gamma^\mu \psi_R A^R_\mu$$

$$- \frac{1}{4} \underbrace{(Z_3 - 1)}_{\equiv \delta_3} F^R_{\mu\nu} F^{\mu\nu}_R + \bar{\psi}_R \Big(i \underbrace{(Z_2 - 1)}_{\equiv \delta_2} \not{\partial} - \underbrace{(Z_2 m_0 - m)}_{\equiv \delta_m = (Z_2 Z_m - 1)\, m} \Big) \psi_R - e \underbrace{(Z_1 - 1)}_{\equiv \delta_1} \bar{\psi}_R \not{A}_R \psi_R$$

2nd line: counter terms

→ "additional" Feynman rules

$$\sim\otimes\sim \; = -i (g^{\mu\nu} p^2 - p^\mu p^\nu)\, \delta_3$$



$-ie\gamma^\mu \delta_1$

$$\leftarrow\otimes- \; = i (\not{p}\, \delta_2 - \delta_m)$$

define $\leftarrow(1PI)- = -i \Sigma(\not{p})$ and $\sim(1PI)\sim = i\Pi^{\mu\nu} \equiv i (g^{\mu\nu} p^2 - p^\mu p^\nu)\, \Pi(p^2)$

then renormalization conditions (in on-shell scheme) are

$\Sigma(\not{p} = m) = 0$ fix pole of fermion propagator @ electron mass

$\left.\frac{d}{d\not{p}} \Sigma(\not{p})\right|_{\not{p}=m} = 0$
$\Pi(q^2 = 0) = 0$ } fix residues of propagator to 1

$-ie\,\Gamma^\mu(q=0) = -ie\gamma^\mu$ $\qquad \Gamma^\mu \equiv$  fix electron charge @ $q=0$ to e

7.3 Dimensional regularization

Regularization ($\neq$ renormalization) : change integrals to make them well defined. At the end of calculation must undo regularization

simply an intermediate step to make expressions well defined

several possibilities to regularize

$$\text{e.g.} \quad \int_0^\infty \frac{d^4k}{(k^2-M^2)^2} \quad \longrightarrow \quad \int_0^{\Lambda_{cut}} \frac{d^4k}{(k^2-M^2)^2}$$

cut off Λ_{cut}, at the end $\Lambda_{cut} \to \infty$

in practice "only" dimensional regularization (DR)

$$\int_0^\infty \frac{d^4k}{(k^2-M^2)^2} \to \int_0^\infty \frac{d^Dk}{(k^2-M^2)^2} \overset{k\to\infty}{\sim} \int \frac{d^Dk}{k^4} \quad \text{convergent for } D<4$$

perform calulation in $D = 4-2\varepsilon$ dimensions, at the end set $D \to 4$ i.e $\varepsilon \to 0$

↯ Warning $\underline{\underline{2}}\,\varepsilon$ is convention !?

No physics associated with $D \neq 4$, just a mathematical procedure

formal requirement for D-dim. integration

- linearity $\quad \int d^Dk \left[f_1(k) + f_2(k)\right] = \int d^Dk\, f_1(k) + \int d^Dk\, f_2(k)$
- scaling $\quad \int d^Dk\, f(k) = s^D \int d^Dk\, f(sk)$
- invariance under shift $\quad \int d^Dk\, f(k) = \int d^Dk\, f(k+q)$

... "usual properties of integration"

advantages of DR : - preserves Lorentz and gauge invariance
- technically relatively simple
- regulates also infrared singularities ($k \to 0$)

Example 1

$$I_1 = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - M^2 + i0^+)^n}$$

<u>use</u> $\Gamma(n) = \int_0^\infty dt\, t^{n-1} e^{-t} = \int_0^\infty d(at)\,(at)^{n-1} e^{-at} = a^n \int_0^\infty dt\, t^{n-1} e^{-at}$

or $\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty dt\, t^{n-1} e^{-t\cdot a}$ Eq. (A)

Note: for $n \in \mathbb{N}$ $\Gamma(n) = (n-1)!$, $\Gamma(n)$ poles for $-n \in \mathbb{N}_0$

<u>use</u> $\int d^Dk\, e^{-k^2} = \int dk_1 e^{-k_1^2} \int dk_2 e^{-k_2^2} \dots \int dk_D e^{-k_D^2} = \pi^{D/2}$ Eq (B)

analytic continuation, $\forall D$!

Then $I_1 = \int \frac{d^Dk}{(2\pi)^D} \frac{(-1)^n}{(-k^2 + M^2 - i0^+)^n} \overset{(A)}{=} \frac{1}{\Gamma(n)} \frac{(-1)^n}{(2\pi)^D} \int_0^\infty dt\, t^{n-1} \int d^Dk\, e^{-t(-k^2+M^2-i0^+)}$

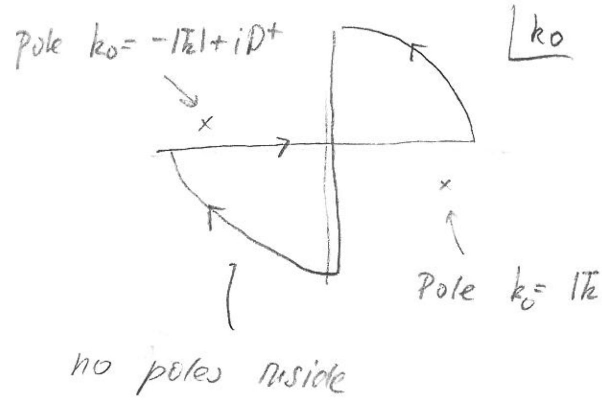
$$= \frac{1}{\Gamma(n)} \frac{(-1)^n}{(2\pi)^D} \int_0^\infty dt\, t^{n-1} e^{-tM^2} \int \frac{d^Dk}{\sqrt{t}^D} e^{+(k^2+i0^+)}$$

change of var $k \to k/\sqrt{t}$

$(k_0^2 - \vec{k}^2 + i0^+) = (k_0 - |\vec{k}| + i0^+)(k_0 + |\vec{k}| - i0^+)$

$D-1$ dim, $0^+ > 0$

<u>Wick rotation</u> (to do k_0 integration)



$$0 = \oint dk_0 = \int dk_0 + \int dk_0 + \int dk_0$$

$$= \int_{-\infty}^{\infty} dk_0 \qquad = 0 \qquad k_0 \to +ik_0$$

ie $\int_{-\infty}^{\infty} dk_0 = +i \int_{-\infty}^{\infty} dk_0\, f(k_0 \to +ik_0)$

$$\Rightarrow I_1 = \frac{i}{\Gamma(n)} \frac{(-1)^n}{(2\pi)^D} \underbrace{\int_0^\infty dt\, t^{n-\frac{D}{2}-1} e^{-t(M^2-i0^+)}}_{\Gamma(n-\frac{D}{2}-1)\,(M^2-i0^+)^{D/2-n}} \underbrace{\int d^Dk\, e^{-k^2}}_{\pi^{D/2}}$$

from Eq (A) (from Eq (B))

thus we get: $$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2+i0^+)^n} = \frac{i(-1)^n}{(4\pi)^{D/2}} \frac{\Gamma(n-D/2)}{\Gamma(n)} (M^2-i0^+)^{D/2-n}$$

correct mass dimension!

Take e.g. $D=4$ and $n=3$, then integral is finite!

Can compute in usual way $d^4k \to dk_0\, d^3\vec{k}$ and will get agreement with result above.

What happens for "divergent" integral e.g. $D=4$ and $n=2$

Dim Reg: $D \to 4-2\varepsilon$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2+i0^+)^2} = \frac{i}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon)(M^2-i0^+)^{-\varepsilon}$$

but $\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)$

Euler constant $\gamma_E = 0.577\ldots$

pole at 0 (or $-n$, $n \in \mathbb{N}_0$)

UV singularity manifest as pole $\frac{1}{\varepsilon}$!

use $(M^2)^{-\varepsilon} = e^{-\varepsilon \ln M^2} = 1 - \varepsilon \ln M^2 + \mathcal{O}(\varepsilon^2)$

$$\Rightarrow \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2+i0^+)^2} = \frac{i}{(4\pi)^2}\Big(\underbrace{\frac{1}{\varepsilon} + \ln(4\pi) - \gamma_E}_{} - \ln(M^2-i0^+) + \mathcal{O}(\varepsilon)\Big)$$

Combination that always appears! $\to \equiv \frac{1}{\bar{\varepsilon}}$

$-i0^+$ needed if $M^2 \in \mathbb{R}$, $M^2<0$

Will not be needed when $D\to 4$, $\varepsilon \to 0$

More examples: $\int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2} = 0$ in DR

($m \to 0$, scaleless integrals

under $k \to s \cdot k$ $\quad I \to s^{D-2} I \quad \Rightarrow I = 0$)

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2+2kp-M^2)^n} = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{((k+p)^2-(p^2+M^2))^n}$$

$$\overset{k\to k-p}{=} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-(p^2+M^2))^n} = \frac{i(-1)^n}{(4\pi)^{D/2}} \frac{\Gamma(n-D/2)}{\Gamma(n)} (p^2+M^2)^{D/2-n}$$

examples of tensor integrals ($k^\mu \dots$ in numerator)

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu}{(k^2-M^2)^n} = 0 \quad \text{(integral is odd under } k^\mu \to -k^\mu)$$

but $$\int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu}{(k^2+2kp-M^2)^n} = \int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu - p^\mu}{(k^2-(p^2+M^2))^n}$$

$$= -p^\mu \cdot \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-(p^2+M^2))^n} \neq 0$$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2-M^2)^n} = g^{\mu\nu} \cdot I_S$$

tensor of rank 2

must be tensor of rank 2 as well!

no p^μ → $g^{\mu\nu}$ only possibility!

find I_S by contraction with $g_{\mu\nu}$

$$I_S = \frac{1}{D} \cdot \int \frac{d^Dk}{(2\pi)^D} \frac{k^2}{(k^2-M^2)^n} = \frac{1}{D} \int \frac{d^Dk}{(2\pi)^D} \left[\frac{1}{(k^2-M^2)^{n-1}} + \frac{M^2}{(k^2-M^2)^n} \right]$$

can generalize this procedure to reduce any 1-loop tensor integral to scalar integrals (Passarino Veltman reduction)

Integrals with more than 1 propagator

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2-m_1^2)((k+p_1)^2-m_2^2)(\cdots)}$$

e.g.  $\rightarrow \frac{1}{(\;)(\;)}$ bubble

 $\rightarrow \frac{1}{(\;)(\;)(\;)}$ triangle

 $\rightarrow \frac{1}{(\;)(\;)(\;)(\;)}$ box

$\rightarrow$ Schwinger parameters / Feynman parameters

$$\frac{1}{a_1^{n_1}\cdots a_m^{n_m}} \overset{\text{Eq. (A)}}{=} \frac{1}{\prod_{i=1}^{m}\Gamma(n_i)} \int_0^\infty \prod_{i=1}^{m}(dt_i\, t_i^{n_i-1})\; e^{-\sum t_i a_i}$$

new variables: $t=\sum t_i$, $t_i=\alpha_i t$ $\qquad$ & $t\cdot\sum\alpha_i = t$

(α_i and t) $\qquad 0\le\alpha_i\le 1 \qquad 0\le t\le\infty$ $\qquad \Rightarrow \sum\alpha_i = 1$

$$= \frac{1}{\prod\Gamma(n_i)} \int_0^\infty dt \int_0^1 \prod_{i=1}^{m}(d\alpha_i\,\alpha_i^{n_i-1})\;\delta(\sum\alpha_i-1)\; t^{\sum n_i-1}\; e^{-t\sum\alpha_i a_i}$$

$\left[\, dt_i\, t_i^{n_i-1} \rightarrow t^{n_i}\, d\alpha_i\, \alpha_i^{n_i-1} \;\;\&\;\; \delta(\sum\alpha_i t - t) = \frac{1}{t}\,\delta(\sum\alpha_i - 1)\right]$

can perform t-integration, using Eq. (A) (α_i: Feynman par.)

$$\frac{1}{a_1^{n_1}\cdots a_m^{n_m}} = \frac{\Gamma(\sum n_i)}{\prod\Gamma(n_i)} \int \prod_{i=1}^{m} d\alpha_i\;\delta(\sum\alpha_i-1)\;\frac{\prod\alpha_i^{n_i-1}}{(\sum\alpha_i a_i)^{\sum n_i}}$$

$\rightarrow$ everything combined to 1 denominator only!

(at the price of introducing $d\alpha_i$ integrations)

In particular:
$$\frac{1}{a_1 a_2} = \int_0^1 d\alpha\, (\alpha\cdot a_1 + (1-\alpha)a_2)^{-2}$$

$$\frac{1}{a_1 a_2 a_3} = 2\int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \left(a_1\alpha + a_2\beta + a_3(1-\alpha-\beta)\right)^{-3}$$

etc...

7.4 The electron self energy

Compute electron propagator at one-loop ($\to Z_2$ & Z_m)

Self energy: $\quad -i\Sigma(p) \equiv \xrightarrow{p}\!\!(\mathrm{1PI})\!\!\rightarrow$

Complete electron propagator (recall Section 5.5)

$$iS_F(p) = \rightarrow + \rightarrow\!\bigcirc\!\rightarrow + \rightarrow\!\bigcirc\!\!-\!\!\bigcirc\!\rightarrow + \dots = iS_F^{(0)}\left(1 - i\Sigma\, iS_F\right)$$

$$iS_F^{(0)} = \frac{i}{\not p - m_0}$$

$$\Rightarrow \quad S_F(p) = \frac{S_F^{(0)}}{1 - S_F^{(0)}\Sigma} \qquad \text{or} \qquad (S_F)^{-1} = (S_F^{(0)})^{-1} - \Sigma(p)$$

Note: we use multiplicative renormalization here!

Compute $-i\Sigma$ at 1-loop:

use Feynman gauge

$$-i\Sigma = \int \frac{d^Dk}{(2\pi)^D} \frac{(-i)(-ie_0)^2\, \gamma^\mu\, i(\not p + \not k + m_0)\, \gamma_\mu}{k^2\left[(p+k)^2 - m_0^2\right]}$$

$$= -e_0^2 \int \frac{d^Dk}{(2\pi)^D} \frac{(2-D)(\not p + \not k) + D m_0}{k^2\left[(k+p)^2 - m_0^2\right]}$$

use Feynman parameter α

$$= -e_0^2 \int_0^1 d\alpha \int \frac{d^Dk}{(2\pi)^D} \frac{(2-D)(\not p + \not k) + D m_0}{\left[\alpha\left[(k+p)^2 - m_0^2\right] + (1-\alpha)k^2\right]^2}$$

$$= \left[(k+\alpha p)^2 + \alpha(1-\alpha)p^2 - \alpha m_0^2\right]^2$$

now shift integration momentum $k \to k - \alpha p$

$$= -e_0^2 \int_0^1 d\alpha \int \frac{d^Dk}{(2\pi)^D} \frac{(2-D)\left(\not k + (1-\alpha)\not p\right) + D m_0}{\left[k^2 - \left(\alpha m_0^2 - \alpha(1-\alpha)p^2\right)\right]^2}$$

$\not k$ term (in numerator) is odd $\to$ vanishes

$\int d^Dk$ now trivial

$$-i\Sigma = \frac{-ie_0^2}{(4\pi)^{D/2}}\,\Gamma(2-D/2)\int_0^1 d\alpha\,\Big((2-D)(1-\alpha)\not p + D m_0\Big)\times\big(\alpha m_0^2-\alpha(1-\alpha)p^2\big)^{\frac{D}{2}-2}$$

expand around $\varepsilon=0$ with $D=4-2\varepsilon$

Note: can only expand integrand, if $\int d\alpha$ after expansion is convergent! (IR singularities can arise!
UV singularity in prefactor $\Gamma(2-D/2)=\Gamma(\varepsilon)$)

$$-i\Sigma = \frac{-ie_0^2}{(4\pi)^2}\left(\frac{1}{\varepsilon}-\gamma_E+\ln(4\pi)\right)\int_0^1 d\alpha\Big[\big(-2(1-\alpha)\not p+4m_0\big)$$

$$+\varepsilon\cdot\Big(2(1-\alpha)\not p-2m_0-(-2(1-\alpha)\not p+4m_0)\cdot\ln(m_0^2\alpha-\alpha(1-\alpha)p^2)\Big)\Big]$$

recall $[e_0]=\varepsilon$ dim full in ??

write this as $e_0^2=\bar{e}_0^2\mu^{2\varepsilon}$ with $[\bar{e}_0]=0$

$$(*)\quad\Rightarrow\ \Sigma(p)=\frac{\bar{e}_0^2}{(4\pi)^2}\left(\frac{1}{\varepsilon}(-\not p+4m_0)+\not p-2m_0+2\int d\alpha\,\big((1-\alpha)\not p-2m_0\big)\ln\left(\frac{m_0^2\alpha-\alpha(1-\alpha)p^2}{\mu^2}\right)+\mathcal{O}(\varepsilon)\right)$$

$$=\frac{\bar\alpha}{4\pi}\left(\frac{1}{\varepsilon}(-\not p+4m_0)+\text{finite}\right)$$

Note: $\underbrace{\langle 0|T(\psi\bar\psi)|0\rangle}_{\text{bare prop}} = Z_2\underbrace{\langle 0|T(\psi_R\bar\psi_R)|0\rangle}_{\text{ren. prop}} \Rightarrow (\text{bare prop})^{-1}=Z_2^{-1}(\text{ren prop})^{-1}$

$$S_F^{-1}=(S_F^{(0)})^{-1}-\Sigma=\not p-m_0-\frac{\alpha}{4\pi}\frac{1}{\varepsilon}(-\not p+4m_0)+\text{finite}$$

$$=Z_2^{-1}\not p-Z_2^{-1}Z_m^{-1}m_0=\left(1+\frac{\alpha}{4\pi}\frac{1}{\varepsilon}\right)\not p-\left(1+\frac{\alpha}{4\pi}\frac{4}{\varepsilon}\right)m_0+\text{finite}$$

$$=\underbrace{\left(1+\frac{\alpha}{4\pi}\frac{1}{\varepsilon}\right)}_{Z_2^{-1}}\left(\not p-\underbrace{\left(1+\frac{\alpha}{4\pi}\frac{3}{\varepsilon}\right)}_{Z_m^{-1}}m_0+\text{finite}\right)+\mathcal{O}(\alpha^2)$$

$$\Rightarrow\quad Z_2=1-\frac{\alpha}{4\pi}\frac{1}{\varepsilon}+\text{finite}+\mathcal{O}(\alpha^2)$$

$$Z_m=1-\frac{\alpha}{4\pi}\frac{3}{\varepsilon}+\text{finite}+\mathcal{O}(\alpha^2)$$

Interlude: do exactly same calculation with different book keeping: renormalized perturbation theory

Now $\quad S_F^{(0)} = \dfrac{i}{\not p - m}$ ← renormalized mass!

but $\quad -i\Sigma = $  $+$  ← counterterms

expressed in m, not m_0!

$$= \underbrace{-i\,\frac{\alpha}{4\pi}\left(\frac{1}{\varepsilon}(-\not p + 4m) + \text{finite}\right)}_{\equiv\, -i\Sigma_2(m)} \underbrace{+\, i\not p\,\delta_2 - i\,\delta_m}_{\text{choose s.t. divergences are absorbed}}$$

$$= i\left(\not p\left(\delta_2 + \frac{\alpha}{4\pi}\frac{1}{\varepsilon}\right) - \left(\delta_m + \frac{\alpha}{4\pi}\frac{4}{\varepsilon}m\right)\right)$$

$$\delta_2 = -\frac{\alpha}{4\pi}\frac{1}{\varepsilon} + \text{finite} = Z_2 - 1 \qquad \delta_m = -\frac{\alpha}{4\pi}\frac{4}{\varepsilon}m + \text{finite} = \boxed{(Z_2 Z_m - 1)\cdot m}$$

renormalization scheme → ren. conditions → fix finite parts

<u>minimal subtraction (MS)</u>

"technical" definition: subtract $\frac{1}{\varepsilon}$ pole and nothing else!

$$Z_2^{MS} = 1 - \frac{\alpha}{4\pi}\frac{1}{\varepsilon}\,; \qquad Z_m^{MS} = 1 - \frac{\alpha}{4\pi}\frac{3}{\varepsilon}$$

<u>modified minimal subtraction ($\overline{MS}$)</u>

subtract $\frac{1}{\varepsilon}$ together with $-\gamma_E + \ln(4\pi)$ i.e. $\frac{1}{\bar\varepsilon}$

$$Z_2^{\overline{MS}} = 1 - \frac{\alpha}{4\pi}\frac{1}{\bar\varepsilon} = 1 - \frac{\alpha}{4\pi}\left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right)$$

$$Z_m^{\overline{MS}} = 1 - \frac{\alpha}{4\pi}\frac{3}{\bar\varepsilon} = 1 - \frac{\alpha}{4\pi}\,3\cdot\left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right)$$

Pole scheme

use renormalized pert. theory:

$$\Sigma(m) = 0 = \Sigma_2(m) \underbrace{- m\,\delta_2 + \delta m}_{= -m(Z_2-1) + m(Z_2 Z_m - 1) \;=\; m\, Z_2\,(Z_m - 1)} = \Sigma_2(m) - (1-Z_m)\cdot m \quad + O(\alpha^2)$$

$$\Sigma_2(m) = \frac{\overline{\alpha}}{4\pi}\left(\frac{1}{\varepsilon}\, 3m - m - 2\int d\alpha\,(1+\alpha)\, m \ln\frac{\alpha^2 m^2}{\mu^2}\right) = \frac{\overline{\alpha}}{4\pi}\, m \left(\frac{3}{\varepsilon} + 4 - 3\ln\frac{m^2}{\mu^2}\right)$$

$$\Rightarrow \quad Z_m = 1 - \frac{\overline{\alpha}}{4\pi}\left(\frac{3}{\varepsilon} + 4 - 3\ln\frac{m^2}{\mu^2}\right)$$

$$\Sigma'(m) = 0 = \Sigma_2'(m) - \delta_2$$

Note: $\Sigma_2'(m) \equiv \frac{d}{d\not p}\Sigma_2(\not p)\big|_{\not p = m}$ has an IR singularity!

cannot use $(*)$, have to differentiate before expansion of integrand in ε!

$$\Sigma_2'(m) = \frac{d}{d\not p}\left(\frac{e_0^2}{(4\pi)^{D/2}}\,\Gamma(2-D/2)\int_0^1 d\alpha\,\left((2-D)(1-\alpha)\not p + D m\right)\times\left(\alpha m^2 - \alpha(1-\alpha)p^2\right)^{\frac{D}{2}-2}\right)\Bigg|_{\not p = m}$$

$$= \frac{e_0^2}{(4\pi)^{D/2}}\,\Gamma(2-\tfrac{D}{2})\, m_0^{D-4}\underbrace{\int_0^1 d\alpha\,(1-\alpha)\,\alpha^{D-5}\left(2(4-D) - (3-D)(2-D)\alpha\right)}_{\frac{D-1}{D-3}}$$

$$= \frac{\overline{\alpha}}{4\pi}\left(\frac{3}{\varepsilon} + 4 - 3\ln\frac{m^2}{\mu^2}\right)$$

$$\Rightarrow \quad Z_2 = 1 - \frac{\overline{\alpha}}{4\pi}\left(\frac{3}{\varepsilon} + 4 - 3\ln\frac{m^2}{\mu^2}\right) \qquad \text{(is equal to } Z_m$$
$\hookrightarrow$ coincidence !)

$$\frac{3}{\varepsilon} = \frac{1}{\varepsilon_{UV}} + \frac{2}{\varepsilon_{IR}}$$

Summary at one-loop:

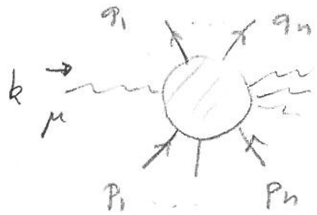
$$m_0 = Z_m^{MS}\, m_{MS} = Z_m^{\overline{MS}}\, m_{\overline{MS}} \quad \Rightarrow \quad m_{\overline{MS}} = m_{MS}\left(1 + \frac{\overline{\alpha}}{4\pi}\left(\ln(4\pi) - \gamma_E\right)\right) + O(\overline{\alpha}^2)$$

$$m_{pole} = Z_m^{-1} Z_m^{\overline{MS}}\, m_{\overline{MS}} = m_{\overline{MS}}\left(1 + \frac{\overline{\alpha}}{4\pi}\left(4 - 3\ln\frac{m^2}{\mu^2}\right)\right) + O(\overline{\alpha}^2)$$

can set $D \to 4$
finite relations

7.6. The Ward - Takahashi Identity

let $\varepsilon_\mu(k) \cdot g^\mu(k; p_1 \dots p_n, q_1 \dots q_n;)$ be a green function with incoming / outgoing fermions, momentum p_i / q_i and a photon with mom. k_μ (plus other photons)

$g(k; p_1 \dots p_n; q_1 \dots q_n) =$ 

m additional photons

not neccessarily on shell!

Let $g_0(p_1 \dots p_n; q_1 \dots q_n)$ be the green function without photon k

We will show: (Ward-Takahashi identity)

$$k_\mu g^\mu(k; p_1 \dots p_n; q_1 \dots q_n) = e \sum_i \left(g_0(p_1 \dots p_n; q_1 \dots q_i - k, \dots) - g_0(p_1 \dots p_i + k \dots; q_1 \dots q_n) \right)$$

For $g \to$ to S-matrix element $\mathcal{M}$ (amputated on-shell green fct)

on L.H.S take multiparticle pole $\prod \frac{i}{\not{p}_i - m} \frac{i}{\not{q}_i - m}$

but R.H.S does NOT have such a pole!

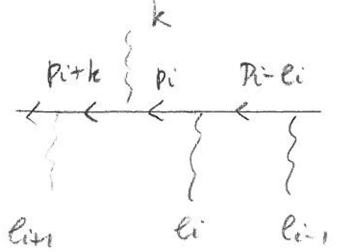
Thus for amplitude for process $\mathcal{M}(k; p_1 \dots p_n; q_1 \dots q_n) = \varepsilon_\mu \mathcal{M}^\mu(\dots)$

we have $\quad k_\mu \mathcal{M}^\mu(k; \{p\}, \{q\}) = 0$

Ward identity : we have used this several times before!

Diagrammatic proof

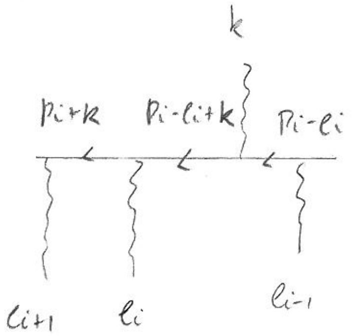
(i) photon k attached to "external" line (for $k_\mu g^{\mu}(\ldots)$)



$$\ldots \frac{i}{\not p_i+\not k-m}(-ie\not k)\frac{i}{\not p_i-m}\gamma^{\nu_i}\frac{i}{\not p_i-\not l_i-m}\ldots$$

$$\not k = (\not p_i+\not k-m)-(\not p_i-m)$$

$$= \ldots\Big(\underbrace{\frac{ie}{\not p_i-m}}_{1st} - \frac{ie}{\not p_i+\not k-m}\Big)\gamma^{\nu_i}\frac{i}{\not p_i-\not l_i-m}$$

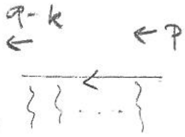


$$= \ldots\frac{i}{\not p_i+\not k-m}\gamma^{\nu_i}\frac{i}{\not p_i-\not l_i+\not k-m}(-ie\not k)\frac{i}{\not p_i-\not l_i-m}\ldots$$

$$= \frac{i}{\not p_i+\not k-m}\gamma^{\nu_i}\Big(\frac{ie}{\not p_i-\not l_i-m} - \underbrace{\frac{ie}{\not p_i-\not l_i+\not k-m}}_{last}\Big)\ldots$$

→ pairwise cancellation of ~~~ terms

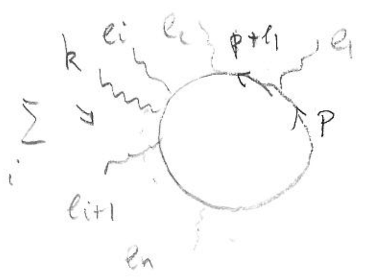
only most left and most right terms survive if we sum over all insertions of the k-photon along fermion line

$\sum$  $= e\cdot\Big($  $-$  $\Big)$

first (most left) last (most right)

This is precisely the statement of the Ward-Takahashi identity

(ii) photon k attached to "internal" fermion loop



again pairwise cancellation as in case (i)

→ we are left with 2 terms again

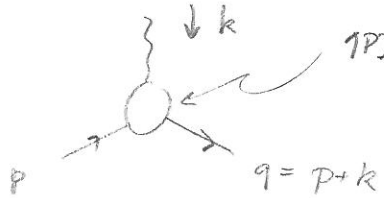
$$k_\mu g^\mu(\ldots) \sim \int \frac{d^D p}{(2\pi)^D}\, e \Big[\mathrm{Tr}\Big(\gamma^{\nu_n} \frac{i}{\not p + \not\ell_1 + \cdots \not\ell_{i+1} - m}\, \gamma^{\nu_{n-1}} \cdots \frac{i}{\not p - m}\Big)$$

$$- \mathrm{Tr}\Big(\gamma^{\nu_n} \frac{i}{\not p + \not\ell_1 + \cdots \not\ell_{i+1} + \not k - m}\, \gamma^{\nu_{n-1}} \cdots \frac{i}{\not p + \not k - m}\Big)\Big]$$

$= 0$ (shift $p \to p+k$ in first term!)

$\Rightarrow$ if photon is attached to internal fermion line (loop) sum over all insertion vanishes $\rightarrow$ completes proof of WTId.

Special case:



1PI part: $-ie\,\Gamma^\mu(p,k) = -ie\,(\gamma^\mu + \Lambda^\mu(p,k))$

higher order corrections

renormalization condition on-shell scheme: $\Gamma^\mu(p, k=0) = Z_1^{-1}\gamma^\mu$

use WTId

$$k^\mu \underbrace{g_\mu(k,p,q)}_{iS_F(p+k)\,(-ie\Gamma^\mu(p,k))\, iS_F(p)} = e\,\Big(\underbrace{g(p, q-k=p)}_{iS_F(p)} - \underbrace{g(p+k=q, q)}_{iS_F(p+k)}\Big)$$

$$\Rightarrow \quad k_\mu \Gamma^\mu(p,k) = S_F^{(-1)}(p+k) - S_F^{-1}(p)$$

$$k_\mu(\gamma^\mu + \Lambda^\mu) = \not p + \not k - \Sigma(p+k) - (\not p - \Sigma(p)) \quad \rightarrow \boxed{k_\mu \Lambda^\mu(p,k) = \Sigma(p) - \Sigma(p+k)}$$

holds to all orders in α!

Note: this implies $\boxed{Z_1 = Z_2}$

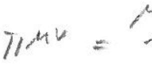
[prop: S_F^{-1} $\quad \not p - m_0 \rightarrow Z_2^{-1}(\not p - m)$

Vertex $\quad \gamma^\mu \rightarrow Z_1^{-1}\gamma^\mu$]

exercise: verify by explicit calculation @ one loop!

7.7. Photon propagator and charge renormalization

$$D_{\mu\nu} \equiv \frac{-i}{p^2}\left(g^{\mu\nu} - (1-\xi_0)\frac{p^\mu p^\nu}{p^2}\right) \quad \text{tree level propagator}$$

photon self energy: $\quad i\Pi^{\mu\nu} =$  $= i(g^{\mu\nu}p^2 - p^\mu p^\nu)\,\Pi(p^2)$

$$\Pi(p^2) = \frac{\alpha_0}{3\pi}\left(-\frac{1}{\varepsilon} + 6\int_0^1 d\alpha\,(1-\alpha)\alpha\,\ln\frac{m_0^2 - \alpha(1-\alpha)p^2}{\mu^2}\right) \quad \text{(exercise)}$$

$\Rightarrow$  $= i\,D_{\mu\rho}\,\Pi^{\rho\sigma}\,D_{\sigma\nu} = \frac{-ig^{\mu\rho}}{p^2}\,i(g^{\rho\sigma}p^2 - p^\rho p^\sigma)\,\frac{-ig^{\sigma\nu}}{p^2}$

terms $\sim (1-\xi_0)$ cancel since $p_\rho\,\Pi^{\rho\sigma} = 0$

$$= \frac{-i}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\cdot\Pi(p^2)$$

$\Rightarrow$  $= \frac{-i}{p^2}\left(g^{\mu\nu}(1+\Pi(p^2)) - (1+\Pi(p^2)-\xi_0)\frac{p^\mu p^\nu}{p^2}\right)$

$$= \frac{-i}{p^2}\underbrace{(1+\Pi(p^2))}_{\equiv Z_3}\left(g^{\mu\nu} - \Big(1 - \underbrace{\frac{\xi_0}{(1+\Pi(p^2))}}_{\equiv\, \xi \,=\, Z_\xi^{-1}\xi_0}\Big)\frac{p^\mu p^\nu}{p^2}\right) \qquad Z_\xi = Z_3\,!$$

$$= Z_3\cdot\frac{-i}{p^2}\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$

from explicit result of $\Pi(p^2)$: $\quad Z_3^{\overline{MS}} = 1 - \frac{\alpha}{3\pi}\frac{1}{\varepsilon}$

Note: If we resum $\quad (1+\Pi(p^2)) \rightarrow \sum_{j=0}^{\infty}(\Pi(p^2))^j = \frac{1}{1-\Pi(p^2)}$

propagator $\sim \frac{-i}{(1-\Pi(p^2))\,p^2}$ $\quad$ pole still at $p^2=0$! no shift!

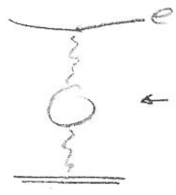
photon remains massless (gauge invariance) and dimensional regularization preserves gauge invariance

Note: finite effects remain after renormalization!

example: (leading term) modification of Coulomb potential

$\rightarrow$ (small) contribution to Lamb shift

split of energy levels $2s_{1/2} - 2p_{1/2}$ in hydrogen



$\leftarrow$ QED effect $\rightarrow$ modification of potential!

tree-level $D_{\mu\nu} \rightarrow \tilde{V}(p) \sim \frac{-i}{p^2} \xrightarrow{\text{Fourier}} \frac{e^2}{4\pi r} \sim V(r)$ use $p^2 = p_0^2 - \vec{p}^{\,2} \simeq -\vec{p}^{\,2}$

non-relativistic limit: $p_0 \sim mv^2$, $\vec{p} \sim mv$, $v \ll 1$

at one-loop $p^2 \ll m^2$

expand $\Pi(p^2)$ for $p^2 \ll m^2$

$$\ln \frac{m^2 - \alpha(1-\alpha)p^2}{\mu^2} = \ln \frac{m^2}{\mu^2} - \alpha(1-\alpha)\frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right)$$

$$\Pi(p^2) = \frac{\bar{\alpha}}{3\pi}\left[-\frac{1}{\varepsilon} + 6\int d\alpha\, \alpha\cdot(1-\alpha)\left(\ln\frac{m^2}{\mu^2} - \alpha(1-\alpha)\frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right)\right)\right]$$

$$= \frac{\bar{\alpha}}{3\pi}\left[-\frac{1}{\varepsilon} + \ln\frac{m^2}{\mu^2} - \frac{1}{5}\frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right)\right]$$

$$\delta\tilde{V}(p) = -\frac{\bar{\alpha}}{3\pi}\frac{1}{5m^2} \xrightarrow{\text{Fourier}} \frac{\alpha}{15\pi m^2}\,\delta(r) \sim V(r)$$

shifts $2s_{1/2}$ level, but not $2p_{1/2}$

Note: this shift is ~ 27 MHz ($> \sim 100\times$ exp error!)

but only small part of total shift (~ 1000 MHz)

renormalization & running of coupling

renormalization requires regularization

$\hookrightarrow$ introduction of (arbitrary) scale

cut-off: Λ , dim reg: μ

renormalized parameters depend on this scale

(dim reg: renormalization scale μ_R)

bare parameters do NOT depend on μ_R

Consider coupling e / α

$$e = e_0 \frac{Z_2 \sqrt{Z_3}}{Z_1} = \underset{\substack{\uparrow \\ \text{Ward id.}}}{\sqrt{Z_3}\, e_0} \qquad \rightarrow \qquad \alpha = Z_3 \alpha_0$$

$$Z_3^{\overline{MS}} = 1 - \frac{\alpha}{3\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

find μ_R dependence:

$$\alpha_0 = \bar{\alpha}\, \mu_R^{2\varepsilon} \left(Z_3^{\overline{MS}}\right)^{-1} \qquad \rightarrow \qquad \bar{\alpha}(\mu_R) = \mu_R^{-2\varepsilon}\, \alpha_0\, Z_3^{\overline{MS}}$$

α_0: dim full, but μ_R independent

$\bar{\alpha}$: dim-less, but μ_R dependent

define β-function (warning: different conventions used)

$$\mu_R \frac{\partial \bar{\alpha}(\mu_R)}{\partial \mu_R} = \beta(\bar{\alpha}) \qquad \left(\text{Sometimes } \mu_R^2 \frac{\partial \bar{\alpha}(\mu_R)}{\partial \mu_R^2} = \beta(\bar{\alpha}) = \frac{1}{2}\mu_R \frac{\partial \bar{\alpha}(\mu_R)}{\partial \mu_R} \right)$$

$\beta(\bar{\alpha})$ determines the "running" (change) of α with μ_R

(initial value needs exp. input! running from theory)

$$\mu_R \frac{\partial \bar{\alpha}}{\partial \mu_R} = -2\varepsilon\, \mu_R^{-2\varepsilon} \alpha_0 \underbrace{\left(1 - \frac{\alpha}{3\pi}\frac{1}{\varepsilon} + O(\alpha^2)\right)}_{Z_3^{\overline{MS}}} \overset{D\to 4}{=} \frac{2\alpha}{3\pi}\alpha_0 + O(\alpha^2) = \frac{2\alpha^2}{3\pi} + O(\alpha^2)$$

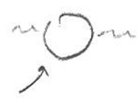
running of coupling: solve differential eq.

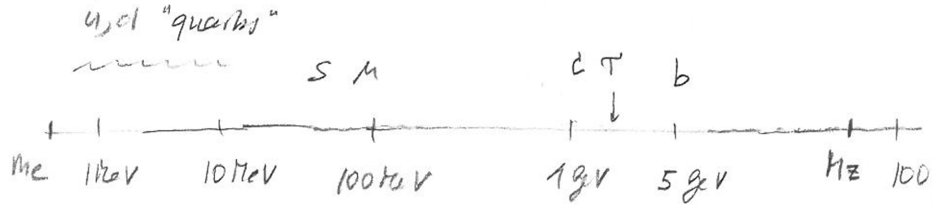
$$\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha(\mu)}{\alpha^2} = \frac{2}{3\pi}\int_{\mu_0}^{\mu} \frac{d\mu_R}{\mu_R} \rightarrow -\frac{1}{\alpha(\mu)} + \frac{1}{\alpha(\mu_0)} = \frac{2}{3\pi}\ln\frac{\mu}{\mu_0}$$

$$\rightarrow \alpha(\mu) = \frac{\alpha(\mu_0)}{1-\frac{2}{3\pi}\alpha(\mu_0)\ln\frac{\mu}{\mu_0}}$$

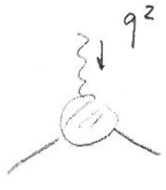
Note: sign of β: positive
→ $\alpha(\mu)$ increases with increasing μ

Landau pole: $1 = \frac{2}{3\pi}\alpha(\mu_0)\ln\frac{\mu}{\mu_0}$ → $\mu = \mu_0\, e^{\frac{3\pi}{2\alpha(\mu_0)}}$ → huge scale
→ well beyond Planck scale

e.g. measure $\alpha(m_e) \simeq \frac{1}{137}$ → running  $\alpha(M_Z) \simeq \frac{1}{128}$



Note: not only coupling, also masses "run"
running governed by "anomalous dimension" (β-function for coupling)

consider  q^2 ← only dim-full quantity

let $p_1^2 = p_2^2 = m^2 \rightarrow 0$

$\bar{\alpha}$: dim-less
naive (classical): $\bar{\alpha}$ cannot depend on q^2 (no other mass scale to make dim-less quant.)

<u>but</u> in QFT: renormalization scale μ_R

$\Rightarrow \bar{\alpha}$ can depend on q^2 via $\frac{q^2}{\mu_R^2}$

$$\alpha(q^2) = \frac{\alpha(q_0^2)}{1-\frac{2}{3\pi}\alpha(q_0^2)\ln\frac{q}{q_0}}$$

resums $\left(\alpha \ln\frac{q}{q_0}\right)^n$
can be important if $q \gg q_0$
(mainly in QCD)

7.8. The magnetic moment

e^- with spin $\rightarrow$ magnetic (dipole) moment $\vec{\mu} = \frac{e}{2m} g \cdot \vec{s}$

g: gyromagnetic factor: classically $g=1$ Dirac $g=2$!

i.e at tree level coupling $\vec{\mu}\cdot\vec{B}$ twice as large as classically expected!

We will (i) Show $g=2$ from Dirac equation

(ii) compute deviation of g from 2 from one-loop contribution : anomalous magnetic moment, $g-2$

(i) non-relativistic limit of Dirac equation

In Dirac representation, write $\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$ ⟸ 2 component spinors

$$\circledast \quad (\not{P} - e\not{A} - m)\begin{pmatrix} \chi \\ \eta \end{pmatrix} = \begin{pmatrix} E - e\phi - m & -(\vec{p}-e\vec{A})\cdot\vec{\sigma} \\ (\vec{p}-e\vec{A})\vec{\sigma} & -E+e\phi-m \end{pmatrix}\begin{pmatrix} \chi \\ \eta \end{pmatrix} = 0$$

2nd line $\circledast$ → $(\vec{p}-e\vec{A})\vec{\sigma}\,\chi = (E - e\phi + m)\,\eta \underset{\text{non-rel. limit!}}{\simeq} 2m\,\eta \quad \Rightarrow \eta = \frac{(\vec{p}-e\vec{A})\cdot\vec{\sigma}}{2m}\chi$

Small components (η) large comp. (χ)

plug into first line of $\circledast$

$$(E - e\phi - m)\chi = (\vec{p}-e\vec{A})\cdot\vec{\sigma}\,\eta = \frac{1}{2m}(\vec{p}-e\vec{A})\cdot\vec{\sigma}\,(\vec{p}-e\vec{A})\cdot\sigma\,\chi$$

(use $\vec{a}\cdot\vec{\sigma}\ \vec{b}\vec{\sigma} = \vec{a}\cdot\vec{b} - i\,\vec{a}\times\vec{b}\cdot\vec{\sigma}$ and $i\vec{p}\times\vec{A} = \vec{\nabla}\times\vec{A} = \vec{B}$)

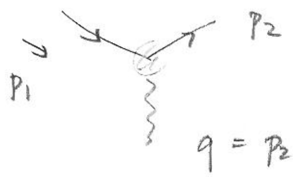
$$= \frac{1}{2m}\left((\vec{p}-e\vec{A})^2 + e\vec{B}\vec{\sigma}\right)\chi$$

$$\Rightarrow \left(\frac{1}{2m}(\vec{p}-e\vec{A})^2 + \underbrace{\frac{e}{2m}\vec{\sigma}\cdot\vec{B}} + e\phi\right)\chi = (E-m)\chi \equiv E_{nr}\,\chi$$

energy without rest mass

$$H = \vec{\mu}\cdot\vec{B} \Rightarrow \vec{\mu} = \frac{e}{2m}\vec{\sigma} = \frac{e}{m}\vec{S} \Rightarrow g = 2$$

(ii) g-2 @ 1 loop



$q = p_2 - p_1$

p_1, p_2 on shell $\quad p_1^2 = p_2^2 = m^2 \quad q p_1 = p_1 p_2 - m^2$

$q p_2 = m^2 - p_1 p_2$

general vertex $\quad \bar{u}(p_2)\, \Gamma^\mu\, u(p_1)$

$\Gamma^\mu = A \cdot \gamma^\mu + B \cdot p_1^\mu + C \cdot p_2^\mu \qquad$ (most general vector !)

Ward id: $\quad q_\mu\, \bar{u}(p_2)\, \Gamma^\mu\, u(p_1) = 0$

$$= \bar{u}(p_2) \Big(\underbrace{A (\not{p}_2 - \not{p}_1)}_{\to 0 \text{ Dirac}} + \underbrace{B\, q \cdot p_1 + C\, q \cdot p_2}_{\Rightarrow B\, q p_1 + C\, q p_2 = 0 \;\Rightarrow\; B = C} \Big)\, u(p_1)$$

$$\Rightarrow \quad \Gamma^\mu = A\, \gamma^\mu + B\, (p_1^\mu + p_2^\mu)$$

rewrite this, using gordon identity

$$\bar{u}(p_2)\, \gamma_\mu\, u(p_1) = \frac{1}{2m}\, \bar{u}(p_2) \left(p_1^\mu + p_2^\mu + i \sigma_{\mu\nu} q^\nu \right) u(p_1)$$

proof $\quad \bar{u}(p_2)\, i\sigma_{\mu\nu} q^\nu\, u(p_1) = -\frac{1}{2}\, \bar{u}(p_2) \left(\gamma^\mu \underbrace{\not{q}}_{\not{p}_2 - m \cdot 1} - \underbrace{\not{q}}_{-\not{p}_1 - m \cdot 1} \gamma^\mu \right) u(p_1)$

$$= -\frac{1}{2}\, \bar{u}(p_2) \left(\gamma^\mu \not{p}_2 + \not{p}_1 \gamma^\mu \right) u(p_1)$$

$$= -\frac{1}{2}\, \bar{u}(p_2) \left(2 p_1^\mu + 2 p_2^\mu - 2 m \gamma^\mu \right) u(p_1)$$

$\gamma^\mu \not{p}_2 = -\not{p}_2 \gamma^\mu + 2 p_2^\mu$ + Dirac

parametrize Γ^μ as follows: $\quad \Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i}{2m} \sigma_{\mu\nu} q^\nu F_2(q^2)$

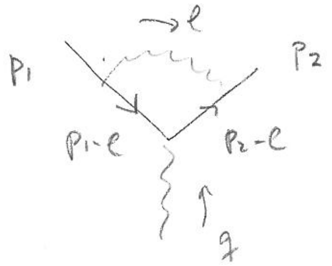
$F_1(q^2)$ and $F_2(q^2)$ form factors $\quad$ (lowest order $F_1 = 1$, $F_2 = 0$)

$F_1 \to$ electric charge

$F_2 \to$ anomalous magnetic moment

$\hookrightarrow$ compute F_2 at one loop! for $q^2 \to 0$

$$(p_1-p_2)^2 = q^2 = 0 = 2m^2 - 2p_1p_2$$



$$\Lambda^\mu = (-ie)^3 (i)^2 (-i) \int \frac{d^D\ell}{(2\pi)^D} \frac{\gamma^\nu (\not{p}_2 - \not{\ell} + m)\gamma^\mu (\not{p}_1 - \not{\ell} + m)\gamma_\nu}{\ell^2 \, (\ell^2 - 2p_1\ell)\,(\ell^2 - 2p_2\ell)}$$

$$= -2e^3 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^D\ell}{(2\pi)^D} \frac{\ldots}{[\alpha(\ell^2 - 2p_1\ell) + \beta(\ell^2 - 2p_2\ell) + (1-\alpha-\beta)\ell^2]^3}$$

$$= -2e^3 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^D\ell}{(2\pi)^D} \frac{\ldots}{[(\ell - \alpha p_1 - \beta p_2)^2 - (\alpha p_1 + \beta p_2)^2]^3}$$

shift $\ell \to \ell + \alpha p_1 + \beta p_2$ $\qquad$ $(\alpha+\beta)^2 m^2$

$$= -2e^3 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^D\ell}{(2\pi)^D} \frac{\gamma^\nu ((1-\beta)\not{p}_2 - \alpha\not{p}_1 - \not{\ell} + m)\gamma^\mu ((1-\alpha)\not{p}_1 - \beta\not{p}_2 - \not{\ell} + m)\gamma_\nu}{[\ell^2 - (\alpha+\beta)^2 m^2]^3}$$

term with $\ell \cdot \ell$ in numerator (divergent)

$$\int \ell^\rho \ell^\sigma \sim g^{\rho\sigma} \to \gamma^\nu \gamma^\rho \gamma^\mu \gamma_\rho \gamma_\nu = (2-D)^2 \gamma^\mu \to F_1 \text{ not } F_2$$

terms with one ℓ in numerator $\to 0$ (odd)

$$\to \Lambda^\mu_{fin} = -2e^3 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \gamma^\nu ((1-\beta)\not{p}_2 - \alpha\not{p}_1 + m)\gamma^\mu ((1-\alpha)\not{p}_1 - \beta\not{p}_2 + m)\gamma_\nu$$

$$\times \underbrace{\int \frac{d^D\ell}{(2\pi)^D} [\ell^2 - (\alpha+\beta)^2 m^2]^3}_{\frac{i(-1)}{(4\pi)^2} \frac{\Gamma(3-2)}{\Gamma(3)} \left((\alpha+\beta)^2 m^2\right)^{-1}}$$

finite! set $D \to 4$

and algebra of numerator in 4 dim using

$$\bar{u}(p_2)\not{p}_2 = \bar{u}(p_2)\, m \quad , \quad \bar{u}(p_2)\not{p}_1 \gamma^\mu \not{p}_2 \, \bar{u}(p_1) = -3m^2\gamma^\mu + 2m(p_1^\mu + p_2^\mu)$$

$$\Lambda^\mu_{fm} = -\frac{e^2}{(4\pi)^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \frac{-4m\left(p_1^\mu(\alpha^2+\alpha\beta-\beta)+p_2^\mu(\beta^2+\alpha\beta-\alpha)\right) + \gamma^\mu \text{ terms}}{m^2(\alpha+\beta)^2}$$

$$= \frac{e^2}{4\pi^2 m}\underbrace{\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \frac{p_1^\mu(\alpha^2+\alpha\beta-\beta)+p_2^\mu(\beta^2+\alpha\beta-\alpha)}{(\alpha+\beta)^2}}_{-\frac{1}{4}(p_1^\mu+p_2^\mu)} + \gamma^\mu \text{ terms}$$

$$\Rightarrow \Lambda^\mu_{fm} = -\frac{e^2}{(4\pi)^2}\,\frac{p_1^\mu+p_2^\mu}{m} + \gamma^\mu \text{ terms}$$

$$= \frac{e^2}{(4\pi)^2}\cdot 2\,\frac{i\sigma_{\mu\nu}q^\nu}{2m} + \gamma^\mu \text{ terms}$$

$$= \frac{\alpha}{2\pi}\,\frac{i}{2m}\,\sigma_{\mu\nu}q^\nu + \gamma^\mu \text{ terms}$$

IR divergent! $\rightarrow F_1(q^2)$

1-loop contribution to $F_2(q^2=0)$ finite!

$$\rightarrow \underbrace{\left(1+\frac{\alpha}{2\pi}+O(\alpha^2)\right)}_{\text{anomalous mag. mom}}\;\underbrace{\frac{i}{2m}}_{g=2}\sigma_{\mu\nu}q^\nu$$

anomalous mag. mom $\quad \hookrightarrow g=2 \quad \Rightarrow \frac{g}{2} = 1+\frac{\alpha}{2\pi}+O(\alpha^2)$

anom. mag mom: $a = \frac{g-2}{2} = 0.00\underbrace{1159652180}$ ...

10 sign. digits agreement exp vs theory!

($\frac{\alpha}{2\pi} = 0.0011614$, need even higher order corrections)

Note: for muon, there is a $\sim 3.5\,\sigma$ "tension" between experiment and theory!