6. QED

6.1. Lagrangian and Feynman rules

$$\mathcal{L}_{QED} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2}_{\text{free photon (+ gauge fixing)}} + \underbrace{\bar\psi(i\partial\!\!\!/ - m_0)\psi}_{\text{free spinor}} \underbrace{- e_0\bar\psi A\!\!\!/\psi}_{\text{interaction}}$$

(gauge invariance, minimal substitution)

m_0: bare mass (parameter in $\mathcal{L}$)

at leading order, S_F has pole @ $p^2 = m_0^2$

beyond leading order, pole is shifted from m_0^2 to m^2

e_0: bare coupling (parameter in $\mathcal{L}$)

Derivation for Feynman rules as in scalar case with some modification:

<u>external legs</u> (for scattering amplitudes)

e.g. photon incoming $\quad A^\mu(x)\,|k_i,\lambda\rangle = A^\mu(x)\,a^+_{k_i\lambda}|0\rangle \quad\rightarrow\quad \varepsilon^\mu(k_i,\lambda)$

coeff of a not 1 but ε^μ

e.g. electron outgoing $\quad \langle p,s|\bar\psi(x) = \langle 0|b_{p,s}\bar\psi(x) \quad\rightarrow\quad \bar u_\alpha(p,s)$

coeff of b^+ not 1, but $\bar u$

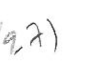
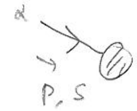
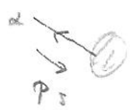
Fermions!

e.g. closed loop $\quad \bar\psi(x_n)\,\psi(x_n)\bar\psi(x_{n-1})\,\ldots\,\psi(x_2)\bar\psi(x_1)\,\psi(x_1) \quad\rightarrow$ factor (-1)

need odd # of anti commutations to get $\psi(x_1)\bar\psi(x_n)$

or relative (-1) factor between diagrams with exchanged fermions in external state

QED: Feynman rules for calculating $iM(2\pi)^4\,\delta(\Sigma p_{in} - \Sigma p_{out})$

- draw all connected & amputated top. distinct diagrams
  relative factor -1 between diagrams obtained by exchanging identical external fermions
- internal photon line  $\underbrace{A^\mu(x) A^\nu(y)}$ $\quad \dfrac{i\left(-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)}{p^2 + i0^+}$
- internal fermion line  $\underbrace{\psi(x)\bar\psi(y)}$ $\quad \dfrac{i(\not p + m_0)_{\alpha\beta}}{p^2 - m_0^2 + i0^+}$
- incoming / outgoing photon  $\varepsilon^\mu(q,\lambda)$  $(\varepsilon^\mu)^*(q,\lambda)$
- incoming / outgoing electron  $u_\alpha(p,s)$  $\bar u_\alpha(p,s)$
- incoming / outgoing positron  $\bar v_\alpha(p,s)$  $v_\alpha(p,s)$
- 4-mom conservation @ each vertex
- integrate over all internal momenta $\int \frac{d^4p}{(2\pi)^4}$

  $\Big\}$ → overall mom cons. + 1 free integration for each loop
- factor (-1) for each closed fermion-loop
- for each vertex  $-ie_0(\gamma^\mu)_{\alpha\beta}$

<u>note</u>: iM is a scalar (complex "number")

→ all spinor and Lorentz indices will be contracted

from u_α, $\dfrac{i(\not p + m_0)_{\alpha\beta}}{p^2 - m_0^2 + i0^+}$ etc

from vertices, ε^μ, γ-propagator

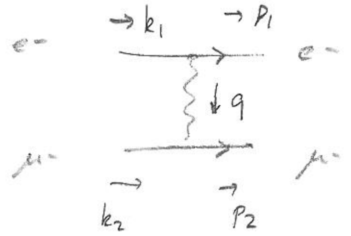
<u>note</u>: fermion propagator is sometimes written as

$$\frac{i}{\not p - m_0} = i(\not p - m_0)^{-1} = i\underbrace{(\not p - m_0)^{-1}(\not p + m_0)^{-1}}_{\frac{1}{p^2 - m_0^2}}(\not p + m_0) = \frac{i(\not p + m_0)}{p^2 - m_0^2}$$

6.2. A sample calculation $e^- \mu^-$ scattering

(for muonic copy of QED with $m_0 \to M_0$)

Only 1 diagram in lowest order $\sim e_0^2$



$$\cdot \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta(k_1 - p_1 - q)\,(2\pi)^4 \delta(k_2 + q - p_2)$$

$$= (2\pi)^4 \delta(k_1 + k_2 - p_1 - p_2) \quad \text{as anticipated!}$$

$\to$ kinematics $\quad q = k_1 - p_1 = p_2 - k_2$

Mandelstam var. $\quad s = (k_1+k_2)^2 = (p_1+p_2)^2 = m^2 + M^2 + 2(k_1 k_2) = \ldots$

$$t \equiv (k_1 - p_1)^2 = (k_2 - p_2)^2 = 2m^2 - 2k_1 p_1 = \ldots$$

M: mass of μ^-

m: mass of e^-

$$u \equiv (k_1 - p_2)^2 = (k_2 - p_1)^2 = m^2 + M^2 - 2k_1 p_2 = \ldots$$

$$s + t + u = 4m^2 + 2M^2 + 2k_1(k_2 - p_1 - p_2) = 2m^2 + 2M^2 = \sum m_i^2$$

$$i\mathcal{M} = (-ie_0)^2\, \bar{u}(p_1)\gamma^\mu u(k_1)\, \frac{-i\left(g_{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right)}{q^2 + i0^+}\, \bar{u}(p_2)\gamma^\nu u(k_2)$$

note: $\bar{u}(p_2)\not{q}\, u(k_2) = \bar{u}(p_2)(\not{p}_2 - \not{k}_2)u(k_2)$
$= 0!$ gauge dep. cancells!

generally: gauge dependence cancels in amplitude, NOT necessarily in single diagram!

$$i\mathcal{M} = \frac{ie_0^2}{q^2}\left(\bar{u}(p_1)_\alpha \gamma^\mu_{\alpha\beta} u(k_1)_\beta\right)\left(\bar{u}(p_2)\gamma_\mu u(k_2)\right) \quad \to \text{scalar!} \quad (-)(X)(|)$$

for cross section need $|\mathcal{M}|^2$

use $\left(\bar{u}(p_1)\gamma_{\mu_1} \cdots \gamma_{\mu_n} u(p_2)\right)^* = \left(u^+(p_1)\gamma^0\gamma_{\mu_1}\cdots\gamma_{\mu_n} u(p_2)\right)^+ \quad \leftarrow * \, T$

$$= u^+(p_2)\gamma^+_{\mu_n}\cdots\gamma^+_{\mu_1}\gamma^0 u(p_1) = u^+(p_2)\gamma^0\gamma_{\mu_n}\cdots\gamma_{\mu_1} u(p_1)$$

$\gamma_0^2 = 1 \quad \& \quad \gamma_0\gamma_\mu^+\gamma_0 = \gamma_\mu$

$$= \bar{u}(p_2)\gamma_{\mu_n}\cdots\gamma_{\mu_1} u(p_1)$$

$$\Rightarrow \quad |M|^2 = \frac{e_0^4}{t^2} \left(\bar{u}_\alpha(p_1) \gamma^\mu_{\alpha\beta} u_\beta(k_1) \cdot \bar{u}_{\alpha'}(k_1) \gamma^{\mu'}_{\alpha'\rho'} u_{\rho'}(p_1) \right) \times \left(\bar{u}(p_2) \gamma_\mu u(k_2) \bar{u}(k_2) \gamma_{\mu'} u(p_2) \right)$$

$$= \underbrace{u_{\rho'}(p_1) \bar{u}_\alpha(p_1)} \gamma^\mu_{\alpha\beta} \underbrace{u_\beta(k_1) \bar{u}_{\alpha'}(k_1)} \gamma^{\mu'}_{\alpha'\rho'}$$

sum over spin $\qquad (\not{p}_1 + m_0)_{\rho'\alpha} \gamma^\mu_{\alpha\beta} (\not{k}_1 + m_0)_{\beta\alpha'} \gamma^{\mu'}_{\alpha'\rho'}$

$$= \mathrm{Tr}\left((\not{p}_1 + m_0) \gamma^\mu (\not{k}_1 + m_0) \gamma^{\mu'} \right) \mathrm{Tr}\left((\not{p}_2 + m_0) \gamma_\mu (\not{k}_2 + m_0) \gamma_{\mu'} \right)$$

trace (in spinor space!)

Spin summed / averaged (over initial states) matrix element squared

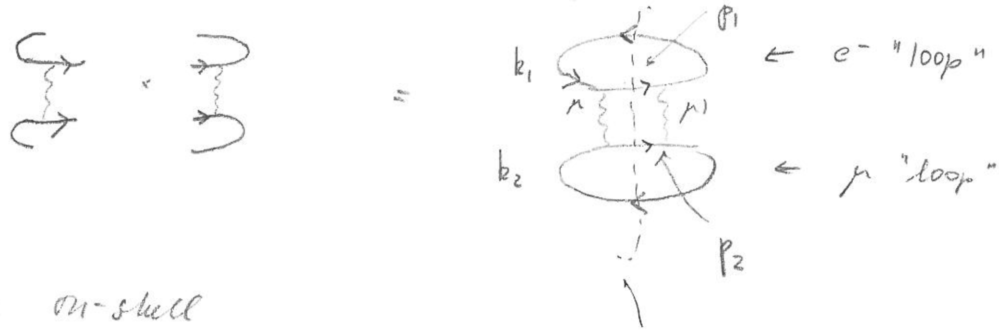
$$\langle |M|^2 \rangle = \frac{e_0^4}{4 \cdot t^2} \mathrm{Tr}\left((\not{p}_1 + m_0) \gamma^\mu (\not{k}_1 + m_0) \gamma^{\mu'} \right) \mathrm{Tr}\left((\not{p}_2 + m_0) \gamma_\mu (\not{k}_2 + m_0) \gamma_{\mu'} \right)$$

$\frac{1}{2} \cdot \frac{1}{2}$ average over initial spin

$\hookrightarrow$ unpolarized cross section

Note: we can read-off traces directly from cut graphs

$M \cdot M^* =$



cut lines: are on-shell

"prop" $\to$ $\delta(p^2 - m_0^2)(\not{p} + m_0)$ (not $\frac{\not{p} + m_0}{p^2 - m_0^2}$) "on-shell" $\qquad$ cut $\to$ cut line on shell

closed fermion loop: trace

Aside: evaluation of traces (in "D-dim" Minkowski space, needed later)

based on $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ → can be generalized to $\underline{D}$ dim

$\{\gamma^\mu, \gamma_5\} = 0$ $\quad \gamma_5^2 = \mathbb{1}$ → going from 4 to D dim problematic

$$\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}) = \frac{1}{2}\left(\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}) + \mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1})\right) = \frac{1}{2}\mathrm{Tr}(\{\gamma^{\mu_1}\gamma^{\mu_2}\})$$

$$= g^{\mu_1\mu_2}\,\mathrm{Tr}(\mathbb{1}) = \underbrace{4}_{\text{dim of spinor space, not } D!} \cdot\, g^{\mu_1\mu_2}$$

even number of γ-matrices

$$\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n}) = \mathrm{Tr}(\gamma^{\mu_2}\ldots\gamma^{\mu_n}\gamma^{\mu_1})$$

$$= 2g^{\mu_1\mu_n}\,\mathrm{Tr}(\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}) - \mathrm{Tr}(\gamma^{\mu_2}\ldots\gamma^{\mu_1}\gamma^{\mu_n}) \qquad \gamma^{\mu_n}\gamma^{\mu_1} = -\gamma^{\mu_1}\gamma^{\mu_n} + 2g^{\mu_1\mu_n}$$

$$= \sum_{i=2}^{n} 2g^{\mu_1\mu_i}\,\mathrm{Tr}(\gamma^{\mu_2}\ldots\cancel{\gamma^{\mu_i}}\ldots\gamma^{\mu_{n-1}}) \underbrace{-}_{(-1)^{n-1} = -1 \quad (n \text{ even})} \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n})$$

$$\Rightarrow \mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) = \sum_{j\neq i} \underbrace{(-1)^{\sigma}}_{\text{sign of permutation}} g^{\mu_j\mu_i}\,\underbrace{\mathrm{Tr}(\ldots\cancel{\gamma^{\mu_j}}\ldots\cancel{\gamma^{\mu_i}}\ldots)}_{\text{trace with } n-2\ \gamma\text{-matrices}}$$

→ recursive relation

odd nr. of γ-matrices

$$\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) = \mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}\gamma_5^2) = -\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma_5\gamma^{\mu_n}\gamma_5)$$

$$= (-1)^n\,\mathrm{Tr}(\gamma_5\gamma^{\mu_1}\ldots\gamma^{\mu_n}\gamma_5) = (-1)\,\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n})$$

$\Rightarrow \mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) = 0$ for n odd further identities with γ-matrices:

- $\gamma^\mu \gamma_\mu = \frac{1}{2}\{\gamma^\mu, \gamma_\mu\} = g^\mu{}_\mu = D \quad (\to 4$ in 4 dimensions$)$

- $\gamma^\mu \gamma_\nu \gamma_\mu = -\gamma^\mu \gamma_\mu \gamma_\nu + \gamma^\mu \{\gamma_\nu \gamma_\mu\} = -D\gamma_\nu + 2g_{\nu\mu}\gamma^\mu = (2-D)\gamma_\nu \quad (\to -2\gamma^\nu$ in 4 dim$)$

- evaluate contraction $\gamma^\mu \gamma_{\nu_1} \cdots \gamma_{\nu_n} \gamma_\mu$ by successive use of $\gamma_{\nu_i}\gamma_\mu = -\gamma_\mu \gamma_{\nu_i} + 2g_{\nu_i \mu}$ to make γ^μ adjacent to γ_μ

use these identities to evaluate

$$\mathrm{Tr}\left((\not p_1 + m_0)\gamma^\mu (\not k_1 + m_0)\gamma^{\mu'}\right) = \mathrm{Tr}\left(\not p_1 \gamma^\mu \not k_1 \gamma^{\mu'}\right) + m_0^2\, \mathrm{Tr}\left(\gamma^\mu \gamma^{\mu'}\right)$$

$$= 4\cdot\left(p_1^\mu k_1^{\mu'} + p_1^{\mu'} k_1^\mu + (p_1 \cdot k_1) g^{\mu\mu'}\right) + 4 m_0^2 g^{\mu\mu'}$$

and multiply by $\mathrm{Tr}\left((\not p_2 + M_0)\gamma_\mu (\not k_2 + M_0)\gamma_{\mu'}\right)$

$$\to \langle |M|^2 \rangle = \frac{2e_0^4}{t^2}\left(s^2 + u^2 - 4(m_0^2 + M_0^2)(s+u) + 6(m_0^2 + M^2)^2\right)$$

t-channel process: exchanged photon $\frac{1}{q^2} \to \frac{1}{t}$

high energy limit $\quad s, |u| \gg M_0^2, m_0^2$

differential cross section $\quad \frac{d\sigma}{d\Omega} = \frac{1}{\text{flux}} \int \langle |M| \rangle^2 \cdot d\Phi \overset{\circledast}{=} \frac{1}{64\pi^2 s} \langle |M| \rangle^2$

exercise

flux $\quad 4\left((k_1 \cdot k_2)^2 - m^2 M^2\right)^{1/2} \sim 2s$

phase space: $(2\pi)^4 \delta(k_1 + k_2 - p_1 - p_2) \prod_{i=1}^{2} \frac{d^3 \vec p_i}{(2\pi)^3 2E_i}$

2 particles in final state

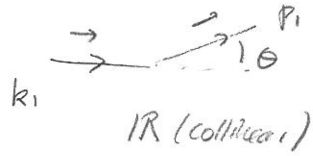
$\circledast$ valid for $2 \to 2$ scattering

$$\frac{d\sigma}{d\Omega} = \frac{e_0^4}{32\pi^2 s} \frac{s^2 + u^2}{t^2}$$

$t = -2p_1 k_1 = -2E_{p_1} E_{k_1}(1 - \cos\theta)$ i.e.

divergence for $t \to 0$

$\hookrightarrow \theta \to 0$



IR (collinear) divergence

6.3 Elementary processes @ tree level

In principle we can now compute any cross section @ tree level i.e. lowest (non-trivial) order in e_0 → no loops

$$d\sigma = \frac{1}{\text{flux}} \int d\Phi \cdot |M|^2$$

$\uparrow$ phase space $\quad d\Phi = \prod_{i=1}^{f} \frac{d^3 \vec{p}_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta(k_1 + k_2 - \sum p_i)$

external photons

$$M = \mathcal{M}^\mu \cdot \varepsilon_\mu^*(k,\lambda) \quad \rightarrow \quad \langle |M| \rangle^2 = \frac{1}{2} \sum_\lambda \mathcal{M}^\mu \mathcal{M}^{*\nu} \varepsilon_\mu^*(k,\lambda) \varepsilon_\nu(k,\lambda)$$

$\uparrow$ incoming photon, mom k, pol. λ $\quad \rightarrow$ need $\sum_{\lambda=1}^{2} \varepsilon_\mu^*(k,\lambda) \varepsilon_\nu(k,\lambda)$

$$\sum_{\lambda=1}^{2} \varepsilon_\mu \varepsilon_\nu^* = \underbrace{-g_{\mu\nu}}_{= \sum_{\lambda=0}^{3}} + \underbrace{\frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n}}_{\text{"subtract unphysical polarizations"}} \equiv \mathcal{g}_{\mu\nu} \quad \left(n^\mu = (k^0, -\vec{k}) \right)$$

note $k^\mu \mathcal{g}_{\mu\nu} = 0$ (consistent with $k^\mu \varepsilon_\mu = 0$)

and $(0, \vec{k})^\mu \mathcal{g}_{\mu\nu} = 0$ (consistent with $\vec{k} \cdot \vec{\varepsilon} = 0$)

in QED: $k_\mu \mathcal{M}^\mu = 0$ (Ward identity)

naive derivation μ $\overset{k}{\rightarrow}$ j_μ (current) but $\partial^\mu j_\mu = 0$ (conserved current)

can replace $\sum_{\lambda=1}^{2} \varepsilon_\mu \varepsilon_\nu^* \rightarrow -g_{\mu\nu}$

- does NOT hold diagram by diagram, only at amplitude level (very similar to gauge parameter ξ cancellation)
- does NOT hold in QCD (non-abelian gauge theory)

Crossing symmetry

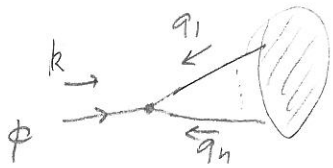
relate processes, where incoming particle is replaced by outgoing antiparticle

recall LSZ: difference between in & out only in sign in Fourier!

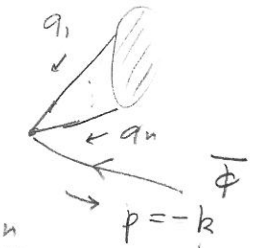
$$\langle \ldots \phi(p) \ldots | S | \ldots \rangle = \langle \ldots | S | \ldots \phi^{+}(k=-p) \ldots \rangle$$

crossing symmetry of S-matrix elements

At diagrammatic level



$$\sum_{i=1}^{n} q_i = -k$$



$$\sum_{i=1}^{n} q_i = p \quad (= -k)$$

diagrams equal except for external leg factor

fermions: $\sum u(k)\bar{u}(k) = \not{k} + m_0 \rightarrow -\sum v(p)\bar{v}(p) = -(\not{p} - m_0) = \not{k} + m_0$

$-$ sign for crossing of fermion

bosons: scalar external factor 1: no change

photon $(\varepsilon^{\mu})^{*} \rightarrow (\varepsilon^{\mu}) \quad \underset{\lambda}{\overset{\Rightarrow}{\sum}} \quad -g^{\mu\nu} \rightarrow -g^{\mu\nu}$ no change

Note: one of the momenta, p or k $(=-p)$ is unphysical (has negative energy)

→ One amplitude can be obtained from the other by analytic continuation!

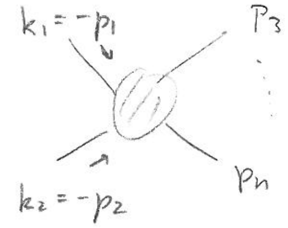
e.g. $\mathcal{M}(\phi(k) \rightarrow \ldots)\Big|_{k_0>0} = \mathcal{M}(\ldots \rightarrow \bar{\phi}(p=-k))\Big|_{\text{analyt. cont to } p_0>0}$

6.4. Helicity method

increase in number of external particles $\to$ tr(..... more γ-matrices)

$\to$ massive increase in number of terms!

$\to$ need more economic techniques



$\to$ helicity method

(particularly useful for massless particles

i.e. high-energy limit, but can be extended to massive fermions)

Do not express $|\mathcal{M}|^2$ in terms of $s_{ij} = (p_i + p_j)^2$ but

express amplitude $\mathcal{M}$ in terms of Weyl spinors

$$|i\pm\rangle \equiv \tfrac{1}{2}(1 \pm \gamma_5)\, u(p_i) \qquad \text{(or } v(p_i)\text{, not distinct in massless case)}$$

$$\langle i\pm| \equiv \bar{u}(p_i)\, \tfrac{1}{2}(1 \mp \gamma_5)$$

helicity 'good' quantum nr. in massless case

spinor products: $\langle ij\rangle \equiv \langle i-|j+\rangle = \bar{u}(p_i)\,\tfrac{1}{2}(1+\gamma_5)\,u(p_j)$

$$[ij] \equiv \langle i+|j-\rangle = \bar{u}(p_i)\,\tfrac{1}{2}(1-\gamma_5)\,u(p_j)$$

note $\langle i-|j-\rangle = \langle i+|j+\rangle = 0$

explicit form (e.g.) $|p+\rangle = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p^-}\, e^{-i\phi(p)} \\ -\sqrt{p^+} \end{pmatrix} \qquad |p-\rangle = \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-}\, e^{i\phi(p)} \\ 0 \\ 0 \end{pmatrix}$

$$p^\pm \equiv E \pm p_z\,, \qquad e^{\pm i\phi(p)} \equiv \frac{p_x \pm i p_y}{\sqrt{p^+ p^-}}$$

properties

$\langle ij\rangle = -\langle ji\rangle\,; \quad [ij] = -[ji] \qquad \langle ij\rangle[ji] = s_{ij} = 2p_i\cdot p_j$

$\langle i-|\gamma^\mu|j-\rangle = \langle j+|\gamma^\mu|i+\rangle \qquad \langle p+|\gamma^\mu|p+\rangle = 2p^\mu \quad \left(= \mathrm{Tr}\left(\tfrac{1}{2}(1+\gamma_5)\,\not{p}\,\gamma^\mu\right)\right)$

$\langle 1-|\gamma^\mu|2-\rangle\langle 3+|\gamma_\mu|4+\rangle = 2\,\langle 14\rangle\,[32]$

$\langle i\pm|\gamma^\mu|j\pm\rangle\,\gamma_\mu = 2\left(|i\mp\rangle\langle j\mp| + |j\pm\rangle\langle i\pm|\right)$

exercise

$\Rightarrow \not{p} = |p+\rangle\langle p+| + |p-\rangle\langle p-|$

$= \sum_{\text{spin}} u\bar{u}(p)$

Schouten identity $\langle 12\rangle\langle 34\rangle = \langle 14\rangle\langle 32\rangle + \langle 13\rangle\langle 24\rangle$

What about external photons, i.e. $\varepsilon_\mu^\pm(p)$ ($\pm$: helicity)

→ express polarization vectors through Weyl spinors

$$\left.\begin{aligned} \varepsilon_\mu^+ &= \varepsilon_\mu^+(p,q) = \frac{\langle p+|\gamma_\mu|q+\rangle}{\sqrt{2}\,\langle qp\rangle} \\ \varepsilon_\mu^- &= \varepsilon_\mu^-(p,q) = \frac{-\langle p-|\gamma^\mu|q-\rangle}{\sqrt{2}\,[qp]} \end{aligned}\right\} \text{ transform as vectors! } \to \text{ section 3!}$$

mom. of photon   reference momentum, remnant of gauge dependence

→ amplitudes must be independent of q

$$p^\mu \varepsilon^\pm_\mu = 0, \quad \varepsilon^\pm_\mu (\varepsilon^{\mu\pm})^* = \varepsilon^\pm_\mu \varepsilon^{\mu\mp} = -1, \quad \sum_{+,-} \varepsilon^\mu(\varepsilon^\nu)^* = -g^{\mu\nu} + \frac{p^\mu q^\nu + p^\nu q^\mu}{p\cdot q}$$

→ proper representation of polarization vectors!

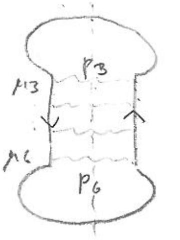
use properties of spinor products:

$$\left.\begin{aligned} \to \quad \not{\varepsilon}^+(p,q) &= \frac{\sqrt{2}}{\langle qp\rangle}\left(|p-\rangle\langle q-| + |q+\rangle\langle p+|\right) \\ \not{\varepsilon}^-(p,q) &= \frac{-\sqrt{2}}{[qp]}\left(|p+\rangle\langle q+| + |q-\rangle\langle p-|\right) \end{aligned}\right\} \text{ exercise}$$

Why is this better?

consider e.g. $e^+(p_1)\, e^-(p_2) \to \gamma(p_3)\,\gamma(p_4)\,\gamma(p_5)\,\gamma(p_6)$

standard technique: $|\mathcal{M}|^2 = \ldots$



$\to \mathrm{tr}\left(\not{p}_1 \gamma^{\mu_3} (\not{p}_1 + \not{p}_3) \gamma^{\mu_2} (\ldots) \ldots\right)$

trace with 16 γ-matrices

→ huge mess

helicity method: $\mathcal{M} = \bar{v}(p_1)\, \not{\varepsilon}_3\, (\not{p}_1+\not{p}_3)\, \not{\varepsilon}_2 \cdots u(p_2)$

→ terms $\langle 1 i\rangle\langle i j\rangle \cdots$

number of terms dramatically reduced

(in particular if # external particles is large)

Independence of reference momentum q (for each external photon)

→ related to gauge invariance

e.g. $\varepsilon^+(p,q_1) - \varepsilon^+(p,q_2) = \frac{1}{\sqrt{2}}\,\frac{1}{\langle q_1 p\rangle\langle q_2 p\rangle}\left(\langle p+|\gamma^\mu|q_1+\rangle\langle q_2 p\rangle - \langle p+|\gamma^\mu|q_2+\rangle\langle q_1 p\rangle\right)$

$$= \frac{1}{\sqrt{2}\,\langle q_1 p\rangle\langle q_2 p\rangle\,[q_1 q_2]}\left(\langle p+|\gamma^\mu \underbrace{\not q_1 \not q_2 |p+\rangle + \langle p+|\gamma^\mu \not q_2 \not q_1}|p+\rangle\right)$$

$$= 2\, q_1\cdot q_2\, \langle p+|\gamma^\mu|p+\rangle = \langle q_1 q_2\rangle[q_2 q_1]\cdot 2p^\mu$$

$$= \frac{-\sqrt{2}\,\langle q_1 q_2\rangle}{\langle q_1 p\rangle\langle q_2 p\rangle}\, p^\mu$$

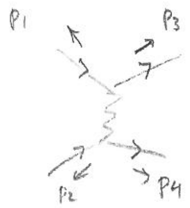
↖ vanishes when contracted with $\mathcal{M}_\mu$

→ clever choice of q simplifies calculation! (e.g. Compton → exercise)

example: $e^-(k_1^+)\,\mu^-(k_2^+) \to e^-(p_3^+)\,\mu^-(p_4^+) \Rightarrow 0 \to e^+(p_1^-)\,\mu^+(p_2^-)\,e^-(p_3^+)\,\mu^-(p_4^+)$

helicity

$= \mathcal{M}(p_1^- p_2^- p_3^+ p_4^+)$



all labelled as outgoing

$p_{1,2} = -k_{1,2}$, helicity flipped

particle ↔ antiparticle

compute $\mathcal{M}(p_1^-, p_2^-, p_3^+, p_4^+) = \frac{(-i)^3}{s_{13}}\,\langle 3+|\gamma^\mu|1+\rangle\langle 4+|\gamma_\mu|2+\rangle$

$$\langle 1-|\gamma^\mu|3-\rangle\langle 4+|\gamma_\mu|2+\rangle = 2\langle 12\rangle[43]$$

$$\Rightarrow \mathcal{M}(p_1^-, p_2^-, p_3^+, p_4^+) = 2\cdot\frac{i\langle 12\rangle[43]}{\langle 13\rangle[31]}\,\frac{\langle 34\rangle}{\langle 34\rangle} = -2i\,\frac{\langle 12\rangle^2[21]}{\langle 13\rangle\underbrace{\langle 4|3|1]}_{=-\langle 42\rangle[21]}} = +2i\,\frac{\langle 12\rangle^2}{\langle 13\rangle\langle 24\rangle}$$

$s_{34} = s_{12}$

Similar $\mathcal{M}(p_1^+, p_2^-, p_3^-, p_4^+) = \frac{(-i)^3}{s_{13}}\,\langle 3-|\gamma^\mu|1-\rangle\langle 4+|\gamma_\mu|2+\rangle = \frac{2i\langle 32\rangle[41]}{\langle 13\rangle[31]}\cdot\frac{\langle 14\rangle}{\langle 14\rangle}$

$[41]\langle 14\rangle = \langle 32\rangle[23]$ ↗ $= \frac{+2i\,\langle 32\rangle^2\,[23]}{\langle 13\rangle\,\langle 4|1|3]} \overset{@}{=} \frac{2i\,\langle 23\rangle^2}{\langle 13\rangle\langle 24\rangle}$

@ $\langle 4|1|3] = \langle 4|-2|3] = -\langle 42\rangle[23]$

- Since helicity conserved along fermion line, only 2 more helicity configuration.
- Changing all helicities $\langle\ \rangle \leftrightarrow [\ ]$ $\equiv$ $\langle i^\pm| \leftrightarrow \langle i^\mp|$ etc

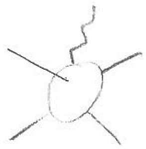
$$\Rightarrow \mathcal{M}(p_1^+ p_2^+ p_3^- p_4^-) = \frac{+2i\,[12]^2}{[13][24]} \qquad \mathcal{M}(p_1^-, p_2^+, p_3^+, p_4^-) = \frac{2i\,[23]^2}{[13][24]}$$

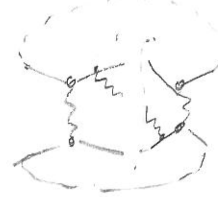
$$\langle |M| \rangle^2 = \frac{e_0^2}{4} \Big(\underbrace{|M(--++)|^2 + |M(+--+)|^2 + |M(++--)|^2 + |M(-++-)|^2}_{} \Big)$$

average   incoherent sum over all (non-vanishing) hel. config

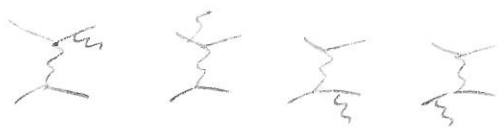
$$= \frac{e_0^2}{4} \cdot 2 \left(4 \frac{s_{12}}{s_{13} s_{24}} + 4 \frac{s_{23}}{s_{13} s_{24}} \right) = 2 e_0^2 \left(\frac{u^2 + s^2}{t^2} \right)$$

→ Can obtain compact results for helicity amplitudes

cp.  conventional vs. helicity method

<u>conventional</u>: e.g. cut graphs  $= Tr(6\text{-gamma}) \times Tr(6\text{-gamma})$

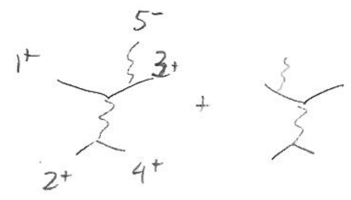
→ $16 = 4 \times 4$ such terms

(4 diagrams)  ⇒ quite large expressions

(mainly due to squaring!)

<u>helicity method</u>

e.g.



$$\simeq \Big(\langle 3+| \varepsilon_5^- (p_5 + p_3) \gamma^\mu |1+\rangle + \langle 3+| \gamma^\mu (p_5 - p_1) \varepsilon_5^- |1+\rangle \Big) \times \langle 4+| \gamma^\mu |2+\rangle$$

$$\varepsilon_5^-(p_5, q) = \frac{-\sqrt{2}}{[q5]} \left(|5+\rangle\langle q+| + |q-\rangle\langle 5-| \right)$$

only 1 term ever contributes!

$$\simeq \Big([3q]\langle 53\rangle \langle 3+|\gamma^\mu|1+\rangle + \langle 3+|\gamma^\mu|5+\rangle [5q]\langle 51\rangle - \langle 3+|\gamma^\mu|1+\rangle [1q]\langle 51\rangle \Big) \times \langle 4+|\gamma^\mu|2+\rangle$$

set $q = p_3$ or $q = p_1$ vanish   ⇝ many terms vanish

- modern methods assemble full amplitudes by "sewing together" "3-point" amplitudes
- standard trace techniques are only used for simpler cases