4. Quantization of free fields

4.1 The real scalar field

recall: for classical field $\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p} \left(a_{\vec{p}}\, e^{-ipx} + a^*_{\vec{p}}\, e^{+ipx} \right)$

$\omega_p = \sqrt{\vec{p}^2 + m^2}$

$p^\mu = (\omega_p, \vec{p})$

with $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \frac{m^2}{2}\phi$

$\leadsto \quad \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}(x)$

	discrete system	continuous system
classical	$q_i(t)$	$\phi(t,\vec{x})$
	$p_i(t)$	$\pi(t,\vec{x})$
$\downarrow$		
quantized	$[\hat{q}_i(t), \hat{p}_j(t)] = i\delta_{ij}$	$[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = i\,\delta(\vec{x}-\vec{y})$
	$[\hat{q}_i(t), \hat{q}_j(t)] = 0$	$[\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] = 0$
	$[\hat{\pi}_i(t), \hat{\pi}_j(t)] = 0$	$[\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = 0$

quantization of fields: classical fields ϕ and π become operators (notation $\phi \to \hat{\phi}$, $\pi \to \hat{\pi}$, ^ will be dropped soon) and we impose equal time commutation relations

Commutation relations between $\hat{\phi}$ and $\hat{\pi}$ reduce

$$[\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{q}}] = (2\pi)^3\, 2\omega_p\, \delta(\vec{p}-\vec{q})$$

$$[\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}] = [\hat{a}^+_{\vec{p}}, \hat{a}^+_{\vec{q}}] = 0$$

$\leadsto$ "copy" of usual ladder operator commutation relation of harm. oscillator, with "continuous" label $\vec{p}$

$\rightarrow$ creation & annihilation operators

$\hat{\phi}$ and $\hat{\pi}$ are operators, but on what Hilbertspace do they act?

recall Section 1.3. : connection KG-field $\leftrightarrow$ harm. oscillator

anticipating interpretation of KG as collection of independent harmonic oscillators, define Fock space $\mathcal{F}$

particles: excitation of modes of field operator

vacuum: $|0\rangle$ def as $\hat{a}_{\vec{p}}\,|0\rangle = 0 \quad \forall \vec{p} \qquad \langle 0|0\rangle = 1$

$$\hat{a}^+_p\,|0\rangle = |p\rangle = |1(\vec{p})\rangle$$

[ ] $\to$ Bose stat

$\hat{a}^+_p\hat{a}^+_q|0\rangle = \hat{a}^+_q\hat{a}^+_p|0\rangle$

general: $$|n_1(\vec{p}_1)\ldots n_m(\vec{p}_m)\rangle = \frac{(\hat{a}^+_{\vec{p}_1})^{n_1}\ldots(\hat{a}^+_{\vec{p}_m})^{n_m}}{\sqrt{n_1!\ldots n_m!}}\,|0\rangle$$

Compute again Hamiltonian, now also an operator in $\mathcal{F}$

$$\hat{H} = \int d^3\vec{x}\,\left(\tfrac{1}{2}\hat{\pi}^2 + \tfrac{1}{2}(\vec{\nabla}\hat{\phi})^2 + \tfrac{m^2}{2}\hat{\phi}^2\right) \qquad \hat{\pi} = \dot{\hat{\phi}}$$

$$\hat{\phi} = \int \widetilde{dp}\,\left(\hat{a}_{\vec{p}}\,e^{-ipx} + \hat{a}^+_{\vec{p}}\,e^{+ipx}\right) \qquad \widetilde{dp} = \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2\omega_p}$$

$$\Rightarrow \hat{H} = \frac{1}{2}\int d^3\vec{x}\int\widetilde{dp}\int\widetilde{dq}\,\Big(\hat{a}_{\vec{p}}\hat{a}_{\vec{q}}\,e^{-i\omega_p t}e^{-i\omega_q t}e^{i\vec{x}(\vec{p}+\vec{q})}\,(-\omega_p\omega_q - \vec{p}\vec{q} + m^2)$$
$$+ \hat{a}^+_{\vec{p}}\hat{a}^+_{\vec{q}}\,e^{+i\omega_p t}e^{+i\omega_q t}e^{-i\vec{x}(\vec{p}+\vec{q})}\,(\;-\;''\;-\;''\;-\;)$$
$$+ \hat{a}_{\vec{p}}\hat{a}^+_{\vec{q}}\,e^{-i\omega_p t}e^{+i\omega_q t}e^{i\vec{x}(\vec{p}-\vec{q})}\,(\omega_p\omega_q + \vec{p}\cdot\vec{q} + m^2)$$
$$+ \hat{a}^+_{\vec{p}}\hat{a}_{\vec{q}}\,e^{i\omega_p t}e^{-i\omega_q t}e^{-i\vec{x}(\vec{p}-\vec{q})}\,(\;-\;''\;-\;''\;-\;)\Big)$$

$$\uparrow\;\hat{\pi}^2 \qquad \uparrow\;(\vec{\nabla}\hat{\phi})^2 \qquad \uparrow\;m^2\phi^2$$

upon $\int d^3\vec{x}$: $\to (2\pi)^3\,\delta(\vec{p}+\vec{q})$ (first two terms), $(2\pi)^3\,\delta(\vec{p}-\vec{q})$ (last two terms)

$$\Rightarrow \frac{1}{2}\int\widetilde{dp}\,\frac{1}{2\omega_p}\Big(\hat{a}_p\hat{a}_{-p}\,e^{-2i\omega_p t}\,(-\omega_p^2 + \vec{p}^2 + m^2)$$
$$\hat{a}^+_{\vec{p}}\hat{a}^+_{-\vec{p}}\,e^{2i\omega_p t}\,(-\omega_p^2 + \vec{p}^2 + m^2) \Big\} \to 0$$
$$+\left(a_p\hat{a}^+_p + \hat{a}^+_p\hat{a}_p\right)\underbrace{(\omega_p^2 + \vec{p}^2 + m^2)}_{2\omega_p^2}\Big)$$

$$\to \hat{H} = \int\widetilde{dp}\,\frac{\omega_p}{2}\left(\hat{a}_p\hat{a}^+_p + \hat{a}^+_p\hat{a}_p\right) \qquad \text{note: } \hat{H}|0\rangle \neq 0\,!$$

subtract (infinite) vacuum energy by normal ordering $:\;:$

put $\hat{a}^+$ to left of $\hat{a}$

$$:\hat{H}: \equiv \int\widetilde{dp}\,\frac{\omega_p}{2}\left(\hat{a}^+_p a_p + \hat{a}^+_p\hat{a}_p\right) = \int\widetilde{dp}\,\omega_p\,\underbrace{\hat{a}^+_p\hat{a}_p}_{\text{counting operator}}$$

Normal ordering is defined as follows (denoted by $: \ldots :$)

operator $\hat{O} = \hat{a}_{p_1} \hat{a}^+_{p_2} \quad \to \quad :\hat{O}: = \hat{a}^+_{p_2} \hat{a}_{p_1}$ & generalised

put all creation operators $\hat{a}^+$ to the left of annihilation op $\hat{a}$

$$:H: |0\rangle = 0$$

$$:H: |n_1(\vec{p}_1) \ldots n_m(\vec{p}_m)\rangle = \sum_{i=1}^{m} n_i \omega_{p_i}$$

---

as $\hat{\phi}$ is an operator it transforms as follows under Poincaré trof.

$$U(\Lambda, a)\, \hat{\phi}(x)\, U^+(\Lambda, a) = \hat{\phi}(\Lambda x + a)$$

with $U(\Lambda, a) = 1 - \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu} + i a_\mu P^\mu$ (for infinitesimal)

in particular, for $\Lambda = \mathbb{1}$ $(\omega_{\mu\nu} = 0)$ and $a_\mu = (t, 0, 0, 0)$

$$e^{itH}\, \hat{\phi}(0, \vec{x})\, e^{-itH} = \hat{\phi}(t, \vec{x}) = \hat{\phi}(x)$$

Schrödinger picture Heisenberg picture (time-dep. operator)

$i \frac{d}{dt} \hat{\phi}(x) = [\hat{\phi}(x), \hat{H}]$ and since $\hat{\phi} = \int d\tilde{q} \left(\hat{a}_q e^{-iqx} + \hat{a}^+_q e^{iqx}\right)$

$\Rightarrow$ ie we need $[\hat{a}_q e^{-i\omega_q t}, \hat{H}] \stackrel{!}{=} \omega_q \hat{a}_q e^{-i\omega_q t}$

$$[\hat{a}^+_q e^{+i\omega_q t}, \hat{H}] = -\omega_q \hat{a}^+_q e^{+i\omega_q t}$$

check: $[\hat{a}_q, \hat{H}] = \int d\tilde{p}\, \omega_p \underbrace{[\hat{a}_q, \hat{a}^+_p]}_{(2\pi)^3 2\omega_q \delta(\vec{p}-\vec{q})} a_p = \omega_q \hat{a}_q$!

$[\hat{a}^+_q, \hat{H}] = \int d\tilde{p}\, \omega_p \hat{a}^+_p \underbrace{[\hat{a}^+_q, \hat{a}_p]}_{-(2\pi)^3 2\omega_q \delta(p-q)} = -\omega_q \hat{a}^+_q$!

equal time commutation relations seem to break L-inv!
(have to choose a frame to quantize theory!)

to verify that quantum theory is independent on the Lorentz frame chosen for quantization, need to show that quantum operator forms of generators still satisfy Poincaré algebra!

$$\left.\begin{aligned} \hat{P}^\mu &\equiv \int d^3\vec{x}\ \hat{T}^{0\mu} = \int \tilde{dp}\ p^\mu \hat{a}^+_{\vec{p}} a_{\vec{p}} \\ \hat{M}^{\mu\nu} &= \int d^3\vec{x}\ (x^\mu \hat{T}^{0\nu} - x^\nu \hat{T}^{0\mu}) \end{aligned}\right\} \text{can check satisfy Poincaré alg.}$$

Furthermore: $[\hat{\phi}(x), \hat{\phi}(y)]$ is L invariant!

Indeed $$[\hat{\phi}(x), \hat{\phi}(y)] = \int \tilde{dp} \int \tilde{dq} \left(\underbrace{[\hat{a}_p, \hat{a}^+_q]}_{(2\pi)^3 2\omega_p \delta(\vec{p}-\vec{q})} e^{-ipx} e^{iqy} + [\hat{a}^+_p, \hat{a}_q] e^{ipx} e^{-iqy} \right)$$

$$= \underbrace{\int \tilde{dp}}_{L\,inv} \left(\underbrace{e^{-ip(x-y)} - e^{ip(x-y)}}_{L\,inv} \right) \equiv i\Delta(x-y)$$

for $x_0 = y_0 = t$, $[\hat{\phi}(x), \hat{\phi}(y)] = [\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] = 0$!

& $\Delta(x-y)$ is L inv $\Rightarrow$ $\boxed{\Delta(x-y) = 0 \text{ for space-like separations } (x-y)^2 < 0}$

As we will now see, this is related to the fact that events with space-like separation cannot affect each other

Considers: $\phi = \int d\tilde{p}\, a_p e^{-ipx} + \int d\tilde{p}\, a_p^\dagger e^{+ipx} = \phi_+ + \phi_-$ (drop ^ now in notation)

"positive frequency part" negative frequency

$$\phi(t,\vec{x})|0\rangle = \int d\tilde{p}\,(\underbrace{a_p e^{-ipx}}_{\to 0} + a_p^\dagger e^{ipx})\,|0\rangle = \underbrace{\int d\tilde{p}\, e^{ipx}\,|p\rangle}_{\text{Fourier trsf. of 1 particle state}} = \phi_-|0\rangle$$

$\phi(t,\vec{x})$ creates particle at time t at point $\vec{x}$

$$\langle 0|\phi(t,\vec{x}) = \langle 0|\int d\tilde{p}\,(a_p e^{-ipx} + \underbrace{a_p^\dagger e^{ipx}}_{\to 0}) = \int d\tilde{p}\,\langle p|e^{-ipx} = \langle 0|\phi_+$$

a particle is annihilated @ $(t,\vec{x})$

$$\Rightarrow \langle 0|\phi(t,\vec{x})|p\rangle = \int d\tilde{p}'\,\langle p'|e^{-ip'x}|p\rangle = \int d\tilde{p}'\, e^{-ip'x}\langle p'|p\rangle$$

but $\langle p'|p\rangle = \langle 0|a_{p'} a_p^\dagger|0\rangle = \langle 0|[a_{p'},a_p^\dagger]|0\rangle = (2\pi)^3 2\omega_p\,\delta(\vec{p}-\vec{p}')\underbrace{\langle 0|0\rangle}_{1}$

thus $\langle 0|\phi(t,\vec{x})|p\rangle = e^{-ipx}$

cp QM $\langle x|p\rangle = e^{i\vec{p}\vec{x}}$

position-space rep. of single particle wave function

Consider $i\Delta_+(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$, "the amplitude" to propagate from y to x, (needs $x_0 > y_0$)

$$i\Delta_+(x-y) = \int d\tilde{p}\int d\tilde{q}\,\langle 0|a_p a_q^\dagger|0\rangle e^{-ipx}e^{iqy} = \int d\tilde{p}\, e^{-ip(x-y)}$$
$$= \langle 0|\phi_+(x)\phi_-(y)|0\rangle$$

similarly $(y_0 > x_0)$

$$-i\Delta_-(x-y) = \langle 0|\phi(y)\phi(x)|0\rangle = \int d\tilde{p}\, e^{+ip(x-y)} = \langle 0|\phi_+(y)\phi_-(x)|0\rangle$$

of course $\Delta_-(x-y) = -\Delta_+(-x+y)$

$\Delta_\pm(x-y)$ fulfill KG equation $(\partial_\mu\partial^\mu + m^2)\Delta_\pm(x-y) = 0$

Propagator: $\quad g(x,y) \equiv i\Delta_F(x-y)$

Green function $\qquad$ Feynman propagator

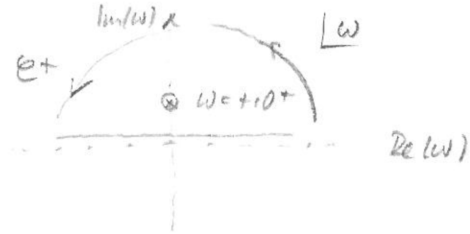
$$i\Delta_F(x) = \theta(x_0)\,\langle 0|\,\phi(x)\,\phi(0)\,|0\rangle + \theta(-x_0)\,\langle 0|\,\phi(0)\,\phi(x)\,|0\rangle$$

$$= \langle 0|\, T\,\phi(x)\,\phi(0)\,|0\rangle$$

Time ordering: put later times to the left

To compute Δ_F, use $\quad \theta(t) = \lim_{0^+ \searrow 0} \int_{-\infty}^{\infty} \frac{d\omega}{2i\pi} \frac{e^{+i\omega t}}{\omega - i0^+} \qquad (0^+ > 0!)$

for $t>0$, use contour C_+

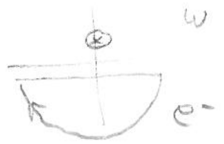


$$e^{i\omega t} = e^{i\operatorname{Re}(\omega)t}\, e^{-\operatorname{Im}(\omega)t}$$

$$\Rightarrow \int_{-\infty}^{\infty} = \int_{C_+} \qquad \text{upper half circle does not contribute}$$

$$\Rightarrow \lim_{0^+ \searrow 0} \int_{-\infty}^{\infty} \frac{d\omega}{2i\pi} \frac{e^{i\omega t}}{\omega - i0^+} = \lim_{0^+ \searrow 0} \int_{C^+} \frac{d\omega}{2i\pi} \frac{e^{i\omega t}}{\omega - i0^+} = \lim_{0^+ \searrow 0} e^{-0^+ t} = 1$$

for $t<0$ use contour C_-



$$\lim_{0^+ \searrow 0} \int_{-\infty}^{\infty} \frac{d\omega}{2i\pi} \frac{e^{+i\omega t}}{\omega - i0^+} = \lim_{0^+ \searrow 0} \int_{C^-} \frac{d\omega}{2i\pi} \frac{e^{i\omega t}}{\omega - i0^+} = 0$$

Notation: "$\lim_{0^+ \searrow 0}$" is often left out

an expression containing 0^+ is always understood to be taken in the limit $0^+ \searrow 0$, $0^+ > 0$!

Now we're ready to compute the propagator:

$$i\Delta_F(x) = \theta(t) \underbrace{\int \widetilde{dp}\, e^{-ipx}}_{\langle 0|\phi(x)\phi(0)|0\rangle = \langle 0|\phi_+(x)\phi_-(0)|0\rangle} + \theta(-t)\int \widetilde{dp}\, e^{+ipx}$$

$$= \lim_{0^+\downarrow 0} \int \widetilde{dp} \int \frac{d\omega}{2i\pi} \frac{1}{\omega - i0^+} \Big(\underbrace{e^{i\omega t} e^{-i\omega_p t}}_{e^{i(\omega-\omega_p)t}} e^{i\vec{p}\cdot\vec{x}} + \underbrace{e^{i\omega(-t)} e^{i\omega_p t}}_{e^{-i(\omega-\omega_p)t}} e^{-i\vec{p}\cdot\vec{x}} \Big)$$

shift int. variables $\quad \omega \to p_0 = \omega_p - \omega \qquad \omega \to p_0 = \omega - \omega_p$

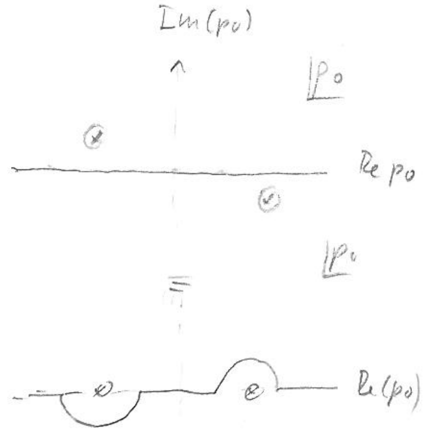
$\vec{p} \to -\vec{p}$

$$= \lim_{0^+\downarrow 0} \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{dp_0}{2i\pi} \frac{1}{2\omega_p} \underbrace{\left(\frac{1}{\omega_p - p_0 - i0^+} + \frac{1}{\omega_p + p_0 - i0^+} \right)}_{\frac{2\omega_p}{\omega_p^2 - p_0^2 - i0^+} = \frac{-2\omega_p}{p^2 - m^2 + i0^+}} e^{-ipx}$$

$$= \left(\lim_{0^+\downarrow 0} \right) \left(+i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2 + i0^+} \right)$$

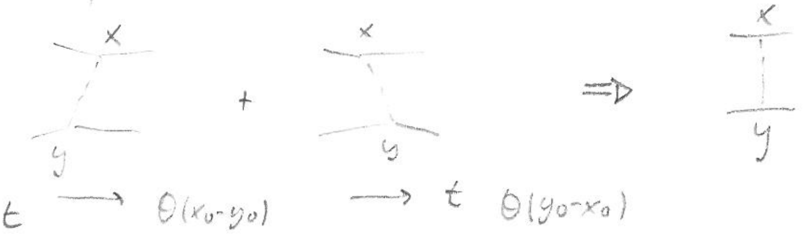
usually omitted in notation

shifts poles in p_0-plane



$$\boxed{\Delta_F = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2 + i0^+}}$$

Feynman propagator sums the two time orderings of (old-fashioned) time ordered perturbation theory!



Note: $(\partial_\mu \partial^\mu + m^2)\Delta_F = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{-p^2+m^2}{p^2-m^2+i0^+} = -\delta(x)$

Δ_F is Green function of Klein-Gordon equation,

4.2. The complex Klein Gordon field

recall Section 1.3. $\mathcal{L} = (\partial_\mu \phi^+)(\partial^\mu \phi) - m^2 \phi^+ \phi$

now $(\hat{\phi}(x) =)$ $\phi(x) = \int \widetilde{dp}\, (a_{\vec p}\, e^{-ipx} + b^+_{\vec p}\, e^{ipx})$

$\phi^+(x) = \int \widetilde{dp}\, (b_{\vec p}\, e^{-ipx} + a^+_{\vec p}\, e^{ipx})$

2 sets of creation & annihilation operators

$[a_{\vec p}, a^+_{\vec q}] = [b_{\vec p}, b^+_{\vec q}] = (2\pi)^3\, 2\omega_p\, \delta(\vec p - \vec q)$

$[a_{\vec p}, b_{\vec q}] = [a_{\vec p}, a^+_{\vec q}] = [a_{\vec p}, b^+_{\vec q}] = [b_p, b_q] = \dots = 0$

now $:H: = \int \widetilde{dp}\, \omega\, (a^+_{\vec p} a_{\vec p} + b^+_{\vec p} b_{\vec p})$

and charge Q from current $j^\mu = i\,(\phi^+ \partial_\mu \phi - (\partial_\mu \phi^+)\phi) = i\phi^+ \overleftrightarrow{\partial}_\mu \phi$

$:Q: = \int d^3\vec x\, :j^0: = \dots = \int \widetilde{dp}\, (a^+_{\vec p} a_{\vec p} - b^+_{\vec p} b_{\vec p})$

classical: source of trouble $\to \rho = j^0$ not pos. def.

here: no problem, 2 kinds of particles (particle & antiparticle)

same mass but opposite charge

$\langle \psi | :Q: | \psi \rangle < 0$ perfectly reasonable, $n_b > n_a$

note: again we "normal order" s.t. $\langle 0 | :Q: | 0 \rangle = 0$

(will drop : : soon in notation)

Feynman propagator: $i\Delta_F(x-y) = \langle 0 | T\, \phi(x)\, \phi^+(y) | 0 \rangle$

for $x_0 > y_0$: $\int \widetilde{dp} \int \widetilde{dq}\, \langle 0 | a_{\vec p}\, a^+_{\vec q} | 0 \rangle\, e^{-ipx} e^{iqy}$ prop. charge +1 from $y \to x$

for $y_0 > x_0$: $\int \widetilde{dp} \int \widetilde{dq}\, \langle 0 | b_{\vec p}\, b^+_{\vec q} | 0 \rangle\, e^{-ipy} e^{iqx}$ prop. charge −1 from $x \to y$

4.3. The Dirac field

$$\mathcal{L} = \bar\psi\,(i\partial\!\!\!/ - m)\,\psi \quad \rightarrow \quad \psi_\alpha(x) = \int \widetilde{dp} \sum_{s=1}^{2} \left(b_{\vec p,s}\, u_\alpha(\vec p,s)\, e^{-ipx} + d^+_{\vec p,s}\, v_\alpha(\vec p,s)\, e^{ipx} \right)$$

$$\bar\psi_\alpha(x) = \int \widetilde{dp} \sum_{s} \left(d_{\vec p,s}\, \bar v_\alpha(\vec p,s)\, e^{-ipx} + b^+_{\vec p,s}\, \bar u_\alpha(\vec p,s)\, e^{ipx} \right)$$

Fermions: states antisymmetric under exchange, i.e.

$$|\vec p,\vec q\rangle = b^+_{\vec p}\, b^+_{\vec q}\, |0\rangle = -|\vec q,\vec p\rangle = -b^+_{\vec q}\, b^+_{\vec p}\, |0\rangle$$

creation & annihilation operators satisfy anticommutation relations

$$\{b_{\vec q,s}\, b^+_{\vec p,r}\} = \{d_{\vec q,s}, d^+_{\vec p,r}\} = (2\pi)^3\, 2\omega_q\, \delta(\vec p-\vec q)\, \delta_{sr}$$

$$\{b_{\vec q}, b_{\vec p}\} = \{d^+_{\vec p}, d^+_{\vec q}\} = \ldots \qquad \{b_{\vec q}, d^+_{\vec q}\} = 0$$

with $\quad \{A,B\} = AB + BA$

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\psi} = i\,\psi^+(x)$$

$$\Rightarrow \quad \{\psi_\alpha(t,\vec x), i\psi^+_\beta(t,\vec y)\} = i\int\widetilde{dp}\int\widetilde{dq} \sum_{s,r} \Big(\{b_{\vec p,s}\, b^+_{\vec q,r}\}\, e^{-ipx}\, e^{iqy}\, u_\alpha(\vec p,s)\, u^+_\beta(\vec q,r)$$

$$+ \{d^+_{\vec p,s}, d_{\vec q,r}\}\, e^{ipx}\, e^{-iqy}\, v_\alpha(\vec p,s)\, v^+_\beta(\vec q,r)\Big)$$

$$= i\int\widetilde{dp}\, \underbrace{\sum_s \Big(e^{i\vec p\cdot(\vec x-\vec y)}\, \underbrace{u_\alpha(\vec p,s)\, u^+_\beta(\vec p,s)}_{(p\!\!\!/+m)\gamma_0} + e^{-i\vec p\cdot(\vec x-\vec y)}\, \underbrace{v_\alpha(\vec p,s)\, v^+_\beta(\vec p,s)}_{(p\!\!\!/-m)\gamma_0}\Big)}$$

change $\int$ var $\vec p \rightarrow -\vec p$

$$= i\int\widetilde{dp}\; 2p_0\, e^{i\vec p(\vec x-\vec y)}\, \mathbb{1}_{\alpha\beta} = i\,\delta(\vec x-\vec y)\,\delta_{\alpha\beta}$$

similar $\quad \{\psi_\alpha(t,\vec x), \psi_\beta(t,\vec y)\} = \{\psi^+_\alpha(t,\vec x), \psi^+_\beta(t,\vec y)\} = 0$

To check whether we could quantize with commutation relations (rather than anti commutation relations) compute H

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = i\psi^+\dot{\psi}$$

$$\Rightarrow \quad H = \int d^3\vec{x}\; i\psi^+\dot{\psi} \qquad \text{(no normal ordering so far)}$$

$$= i\int d^3\vec{x}\int d\tilde{p}\int d\tilde{q}\sum_{r,s}\left(d_{\vec{p},r}\, v^+(\vec{p},r)\, e^{-ipx} + b^+_{\vec{p},r}\, u^+(\vec{p},r)\, e^{+ipx}\right)\cdot$$
$$\left(-i\omega_q\, b_{\vec{q},s}\, u(\vec{q},s)\, e^{-iqx} + i\omega_q\, d^+_{\vec{q},s}\, v(\vec{q},s)\, e^{iqx}\right)$$

use $u^+u = v^+v = 2E$, and proceed as in scalar case

$\int d^3\vec{x} \;\rightarrow\; \delta(\vec{p}-\vec{q}) \;\rightarrow\; \int d^3\vec{q}$

$$H = \int d\tilde{p}\;\omega_p\sum_s\left(b^+_{\vec{p},s}\, b_{p,s} - d_{\vec{p},s}\, d^+_{\vec{p},s}\right)$$

So far, no commutation relations have been used!

If we were to use commutation relation, then

$$?? \quad :H: = \int d\tilde{p}\;\omega_p\sum_s\left(b^+_{\vec{p},s}\, b_{p,s} - d^+_{\vec{p},s}\, d_{p,s}\right) \quad ??$$

<u>Not</u> positive definite!

$:H:$ not bounded from below?

$\rightarrow$ use anti-commutation relations

$$:d_{\vec{p},s}\, d^+_{\vec{p},s}: \;=\; -\,d^+_{\vec{p},s}\, d_{p,s}$$

$$\Rightarrow \quad :H: = \int d\tilde{p}\;\omega_p\sum_s\left(b^+_{\vec{p},s}\, b_{p,s} + d^+_{\vec{p}s}\, d_{\vec{p},s}\right)$$

$\rightarrow$ see Peskin & Schröder for more detailed discussion

need to quantize spin ½ using anti-commutation relations

Propagator $\quad S_F(x-y) = \int\frac{d^4p}{(2\pi)^4}\,\frac{\not{p}+m}{p^2-m^2+i0^+}\,e^{-ip(x-y)} \qquad$ (exercise)

4.4. The electromagnetic field

The Lagrangian for the free em field A^μ is

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ field-strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

gauge invariance: $A_\mu \to A_\mu + \partial_\mu \chi$, $\chi(x)$ arbitrary function

$\to$ does not change $\mathcal{L}$ (local symmetry)

use this freedom to put constraints on A_μ (fixing the gauge)

- Lorentz gauge $\partial_\mu A^\mu = 0$

  not unique, can still perform $A^\mu \to A^\mu + \partial^\mu \chi$ if $\partial_\mu \partial^\mu \chi = 0$

- Coulomb gauge (radiation gauge) $\vec{\nabla} \cdot \vec{A} = 0$ (not covariant)

  Maxwell: $\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\mu$ use $\vec{\nabla}\cdot\vec{A} = 0$ and $\nu = 0 \Rightarrow -\vec{\nabla}^2 A^0 = j^0$

  Integr. $\to$ $A^0 = \int \frac{d^3y}{4\pi |\vec{x}-\vec{y}|} j^0$ $\to A^0$ not independent ($=0$ in free case)

  $\hookrightarrow$ only two physical d.o.f. present

e.g. wave along $\vec{p} = (0, 0, p_z)$

$$\underbrace{\varepsilon^\mu(k,\lambda=0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \varepsilon^\mu(k,\lambda=3) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{\text{unphysical}} \qquad \varepsilon^\mu(k,\lambda=1) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \varepsilon^\mu(k,\lambda=2) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \text{linear pol}$$

$$\varepsilon^\mu(k,+) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix} \quad \varepsilon^\mu(k,-) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\i\\0\end{pmatrix} \quad \text{circular pol.}$$

2 options for quantization

(i) drop covariance, quantize only the two physical d.o.f.

(ii) keep covariance, but fix gauge

note conj. mom $\Pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_t A^0)} = 0$!?

$\Big\}$ $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)$

$\to$ have to deal with non-physical polarizations

Quantization in Coulomb gauge

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \rightarrow \quad \pi^i = \frac{\partial \mathcal{L}}{\partial(\dot{A}_i)} = -\dot{A}^i - \partial^i A^0 = E^i$$

want to impose $[A_i(t,\vec{x}), E^j(t,\vec{y})] = i\,\delta_{ij}\,\delta(\vec{x}-\vec{y})$

but this is inconsistent with $\vec{\nabla}\cdot\vec{A} = 0$

$$[\vec{\nabla}\cdot\vec{A}, E^j] = i\,\partial_j\,\delta(\vec{x}-\vec{y}) \neq 0 \quad ⚡$$

r.h.s. of commutation relations will be modified!

start from $[a_{\vec{p},\lambda}, a^+_{\vec{q},\lambda'}] = (2\pi)^3\, 2\omega_q\, \delta(\vec{p}-\vec{q})\,\delta_{\lambda\lambda'}$ $\qquad \lambda,\lambda' \in \{1,2\}$

and $\vec{A}(x) = \int d\tilde{p} \sum_{\lambda=1}^{2} \left(a_{\vec{p}\lambda}\, \vec{\varepsilon}(p,\lambda)\, e^{-ipx} + a^+_{\vec{p}\lambda}\, \vec{\varepsilon}^*(p,\lambda)\, e^{+ipx} \right)$

2 phys. pol. $\qquad \vec{p}\cdot\vec{\varepsilon} = 0$ (Coulomb gauge)

$\Rightarrow \quad H = \int d\tilde{p}\; \omega_p \sum_{\lambda=1}^{2} a^+_{\vec{p}\lambda}\, a_{\vec{p}\lambda}$ $\qquad$ (after normal ordering)

but $[A_i(t,\vec{x}), E^j(t,\vec{y})] = i \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\delta^{ij} - \frac{p^i p^j}{\vec{p}^2} \right) e^{i\vec{p}(\vec{x}-\vec{y})} = i\, \delta^{tr}_{ij}(\vec{x}-\vec{y})$

not covariant! $\qquad \partial_i\, \delta^{tr}_{ij}(\vec{x}-\vec{y}) = 0$!

propagator is also not covariant in Coulomb gauge

$$\langle 0 | T A_j(x) A_i(y) | 0 \rangle = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i0^+} \cdot \underbrace{\sum_{\lambda=1}^{2} \varepsilon^\mu(p,\lambda) \left(\varepsilon^\nu(p,\lambda)\right)^*}_{\left(\delta_{ij} - \frac{p_i p_j}{\vec{p}^2}\right)}$$

$$\hookrightarrow \langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i0^+} \left(\underbrace{-g_{\mu\nu}}_{\text{covariant part}} + \underbrace{\text{additional terms}}_{\text{will not contribute to physical observables}} \right)$$

Quantisation in Lorentz gauge

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \quad \Rightarrow \quad \begin{aligned} \Pi^\mu &= F^{\mu 0} - \frac{1}{\xi} g^{\mu 0} (\partial_\nu A^\nu) \\ \Pi^0 &= -\frac{1}{\xi} \partial_\mu A^\mu \neq 0 \end{aligned}$$

e.o.m. $\partial_\nu \partial^\nu A^\mu - (1 - \frac{1}{\xi}) \partial^\mu \partial_\nu A^\nu = 0$

gauge freedom
↳ "too many" d.o.f
↳ "over quantization"

reduce gauge freedom (not completely!) in covariant way

$\mathcal{L}$ now gauge dependent (arbitrary parameter ξ)

gauge fixing → $\Pi^0 \neq 0$, try to impose covariant commutation relations

$$[A_\mu(t,\vec{x}), \Pi_\nu(t,\vec{y})] = i g_{\mu\nu} \delta(\vec{x} - \vec{y})$$

$$[A_\mu(t,\vec{x}), A_\nu(t,\vec{y})] = [\Pi_\mu(t,\vec{x}), \Pi_\nu(t,\vec{y})] = 0$$

this requires that the Lorentz gauge condition $\partial_\mu A^\mu = 0$ does <u>not</u> hold as operator identity, but only in weak sense, ie. $\langle \psi | \partial_\mu A^\mu | \psi \rangle = 0$ for <u>physical</u> states $|\psi\rangle$

(*) $\Rightarrow [a_{\vec{p},\lambda}, a^\dagger_{\vec{q},\lambda'}] = -g_{\lambda\lambda'} 2\omega_p (2\pi)^3 \delta(\vec{p} - \vec{q})$, $\quad \lambda, \lambda' \in \{0,1,2,3\}$

and $A^\mu(x) = \int \widetilde{dp} \sum_{\lambda=0}^{3} \left(a_{\vec{p}\lambda} \varepsilon(\vec{p},\lambda) e^{-ipx} + a^\dagger_{\vec{p}\lambda} \varepsilon^*(\vec{p},\lambda) e^{+ipx} \right)$

sum over all 4 polarization, also unphysical ones

$\Rightarrow \quad H = \int \widetilde{dp}\, \omega_p \left(\sum_{\lambda=1}^{3} a^\dagger_{\vec{p}\lambda} a_{\vec{p}\lambda} - a^\dagger_{\vec{p}0} a_{p0} \right)$ (after normal ordering)

states with negative energy?!

also "wrong" sign in (*) results in negative norm states

$|\psi\rangle \equiv \int \widetilde{dp}\, f(p)\, a^\dagger_{\vec{p}0} |0\rangle = \int \widetilde{dp}\, f(p) |p_{\lambda=0}\rangle$

$\Rightarrow \quad \langle \psi | \psi \rangle = \int \widetilde{dp} \int \widetilde{dq}\, f(p) f^*(q) \langle 0 | a_{\vec{q}0} a^\dagger_{\vec{p}0} | 0 \rangle = -\int \widetilde{dp}\, |f(p)|^2 < 0$ ↯

2. Gupta Bleuler formalism

define "physical states" as states satisfying $(\partial_\mu A^\mu)_+ |\Psi_{phys}\rangle = 0$

take pos. frequency part of A_μ (annih. op)

$\Rightarrow \quad \langle \Psi_{phys} | (\partial_\mu A^\mu)_- = 0$

$\rightarrow \quad \underbrace{\langle \Psi'_{phys} | \partial_\mu A^\mu | \Psi_{phys}\rangle}$ $= \langle \Psi'_{phys} | (\partial_\mu A^\mu)_- + (\partial^\mu A^\mu)_+ |\Psi_{phys}\rangle = 0$

Lorentz gauge constraint in weak form
not as operator identity $\partial_\mu A^\mu = 0$

consider general 1-particle state. wlog set $p^\mu = (E,0,0,E)$

$$|\Psi\rangle = \sum_{\lambda=0}^{3} c_\lambda \, a^+_{\vec p,\lambda} |0\rangle \quad \in \mathcal{H}_1$$

arbitrary coefficient

subspace of transverse states: $|\Psi_T\rangle = \sum_{\lambda=1}^{2} c_\lambda \, a^+_{\vec p,\lambda} |0\rangle$

these are physical states!

$$(\partial_\mu A^\mu)_+ |\Psi_T\rangle = \int d\tilde q \sum_{\lambda'=0}^{3} q\cdot\varepsilon(q,\lambda') \, e^{-iqx} \, a_{\vec q,\lambda'} \sum_{\lambda=1}^{2} c_\lambda \, a^+_{\vec p,\lambda} |0\rangle$$

$\lambda' = 0,3: \quad a_{\vec q,\lambda'} \, a^+_{p,\lambda} |0\rangle = a^+_{p,\lambda} \, a_{\vec q,\lambda'} |0\rangle = 0$

$\lambda' = 1,2 \quad \int d\tilde q \; q\cdot\varepsilon(q,\lambda') \, a_{\vec q\lambda'} \, a^+_{\vec p\lambda} |0\rangle = p\cdot\varepsilon(p,\lambda)|0\rangle = 0$ for $\lambda \in \{1,2\}$

$|\Psi_{SL}\rangle \equiv (a^+_{\vec p,0} - a^+_{\vec p,3})|0\rangle$ is also a physical state!

$$(\partial_\mu A^\mu)_+ |\Psi_{SL}\rangle = \left(p\cdot\varepsilon(p,0) - p\cdot\varepsilon(p,3) \right)|0\rangle = (E-E)|0\rangle = 0$$

The subspace of $\mathcal{H}_1$ spanned by $|\Psi_T\rangle$ and $|\Psi_{SL}\rangle$ is the "physical" 1 particle space

scalar-long. "mix"

We have an equivalence relation

$$|\psi_T\rangle \sim |\psi_T'\rangle = |\psi_T\rangle + \kappa\, |\psi_{SL}\rangle$$

κ ← arbitrary coefficient

Can show (exercise)

$$\langle \psi_T' | \psi_T' \rangle = \langle \psi_T | \psi_T \rangle \quad > 0$$

$$\langle \psi_T' | P^\mu | \psi_T' \rangle = \langle \psi_T | P^\mu | \psi_T \rangle \quad , \quad \langle \psi_T' | H | \psi_T' \rangle \geq 0$$

$$\langle \psi_{phys} | \psi_T' \rangle = \langle \psi_{phys} | \psi_T \rangle$$

Identify photons as equivalence class w.r.t. this equivalence relation. Can work with any representative of class (i.e. any value of κ). Most convenient choice: $\kappa = 0$

→ purely transverse states!

Propagator $iD^{\mu\nu}(x-y) \equiv \langle 0 | T\, A^\mu(x)\, A^\nu(y) | 0 \rangle$ $\left(\text{for } \mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \right)$

$$D^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i0^+} \left(-g^{\mu\nu} + (1-\xi) \frac{p^\mu p^\nu}{p^2} \right)$$

(→ $D^{\mu\nu}$ is Green function of ξ-dep. e.o.m. → exercise)
propagator is gauge dependent, but physical observables must (and will) be gauge independent! ξ-dep. cancels!

doing calculation: either keep ξ arbitrary and check ξ-cancellation

pick gauge from beginning: $\xi = 1$: Feynman gauge

$\xi = 0$ Landau gauge

Note: Gupta Bleuler quantization works for QED (abelian gauge theory) but for non-abelian case (QCD, electroweak) we will need more powerful techniques (→ path integrals)

4.5, C, P and T

Under parity, a state $|\vec{p}, s; a\rangle \equiv a^+_{\vec{p},s}|0\rangle$ transforms as

$$P|\vec{p}, s; a\rangle = \eta_a |-\vec{p}, s; a\rangle$$

↑ phase: intrinsic parity of particle a

P: unitary operator $PP^+ = 1$, but also $P^2 = 1 \rightarrow P = P^+$

$$\rightarrow \quad P a^{(+)}_{\vec{p},s} P = \eta_a a^{(+)}_{-\vec{p},s}$$

spin ½ field:

$$P\psi(t,\vec{x})P = \int d\tilde{p} \sum_s \left(\eta_b b_{-\vec{p},s} u(\vec{p},s) e^{-ipx} + \eta_d d^+_{-\vec{p},s} v(\vec{p},s) e^{ipx}\right)$$

$$\vec{p} \rightarrow -\vec{p} \quad = \int d\tilde{p} \sum_s \left(\eta_b b_{\vec{p},s} u(-\vec{p},s) e^{-ip\tilde{x}} + \eta_d d^+_{\vec{p},s} v(-\vec{p},s) e^{ip\tilde{x}}\right)$$

with $\tilde{x} = (t, -\vec{x})$

next use $u(-\vec{p},s) = \gamma_0 u(\vec{p},s)$ and $v(-\vec{p},s) = -\gamma_0 v(\vec{p},s)$

$$P\psi(t,\vec{x})P = \int d\tilde{p} \sum_s \left(\eta_b b_{\vec{p},s} \gamma_0 u(\vec{p},s) e^{-ip\tilde{x}} - \eta_d d^+_{\vec{p},s} v(\vec{p},s) e^{ip\tilde{x}}\right)$$

$$\stackrel{!}{=} \eta_b \gamma_0 \psi(t, -\vec{x}) \qquad \text{requires } \underbrace{\eta_d = -\eta_b}$$

intrinsic parity of e^- is opposite to e^+

for complex scalar:

$$P\phi(t,\vec{x})P = \int d\tilde{p} \left(\eta_a a_{\vec{p}} e^{-ip\tilde{x}} + \eta_b b^+_{\vec{p}} e^{ip\tilde{x}}\right) = \eta_a \phi(t, -\vec{x})$$

$\eta_a = \eta_b$! same parity for particle & antiparticle

Note: for scalars $\eta^2 = 1 \Rightarrow \eta = \pm 1$

for fermions $\eta^2 = \pm 1$ (expressions always bilinear in ψ !)

except for Majorana fermions $\eta^2 = +1 \rightarrow \eta = \pm 1$

for vector field: $P\vec{A}(t,\vec{x})P = -\vec{A}(t,-\vec{x})$

"true" vector field

→ photon: $P|\vec{p},s;\gamma\rangle = -|-\vec{p},s;\gamma\rangle$ intrinsic parity -1

Charge conjugation

$C b^{(+)}_{\vec{p},s} C = d^{(+)}_{\vec{p},s}$ and $C d^{(+)}_{\vec{p},s} C = b^{(+)}_{\vec{p},s}$ (ignore possible phases)

$C\psi(x)C = i\gamma^2\psi^*(x)$ (see Section 3)

photon field: $C A^\mu(x) C = -A^\mu(x)$

$\Updownarrow$

$C a^{(+)}_{\vec{p},\lambda} C = -a^{(+)}_{\vec{p},\lambda}$ photon: charge conj. -1

Time reversal (antilinear operator)

$T b^{(+)}_{\vec{p},s} T = b_{-\vec{p},-s} \equiv (b_{-\vec{p},2}, -b_{\vec{p},1})$

from requirement that $T\psi T$ satisfies t-reversed Dirac eq.

$T\psi(t,\vec{x})T = i\gamma^1\gamma^3\psi(-t,\vec{x})$ (as in Section 3.)

Summary (CPT theorem)

	$\bar\psi\psi$	$i\bar\psi\gamma_5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma_5\psi$
P	+1	−1	$(+1,-\vec{1})$	$(-1,+\vec{1})$
C	+1	−1	$(+1,-\vec{1})$	$(+1,-\vec{1})$
T	+1	+1	(−1)	+1
CPT	+1	+1	−1	−1

$\bar\psi\sigma^{\mu\nu}\psi \xrightarrow{CPT} +\bar\psi\sigma^{\mu\nu}\psi$

$\underbrace{\text{CPT: } +1 \quad +1}_{\text{invariant under CPT}}$ $\underbrace{-1 \quad -1}_{\text{needs to be contracted with vector!}}$

($\partial_\mu \to -\partial_\mu$ under CPT)

→ CPT theorem: cannot write a Lorentz scalar that is not invariant under CPT (with hermitian H)

P, C and CP=T are "easy" to violate