3 Dirac equation

3.1 Weyl equation

We know that $\psi_R \to \psi_R' = D_R(\Lambda)\,\psi_R \qquad D_R = e^{(-i\vec\theta+\vec\eta)\frac{\vec\sigma}{2}}$

for $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu \qquad \omega^\mu{}_\nu = \begin{pmatrix} 0 & \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & 0 & -\theta_3 & \theta_2 \\ \eta_2 & & 0 & -\theta_1 \\ \eta_3 & \ominus & & 0 \end{pmatrix}$

claim: $\psi_R^+ \sigma^\mu \psi_R$ transforms like a 4-vector $\to \Lambda^\mu{}_\nu\, \psi_R^+ \sigma^\nu \psi_R$

def: $\sigma^\mu \equiv (1, \vec\sigma) = (\sigma^0, \vec\sigma)$

proof: (exercise)

$$\psi_R^+ \begin{pmatrix} \sigma^0 = 1 \\ \vec\sigma \end{pmatrix} \psi_R \to \psi_R^+\, e^{(+i\vec\theta+\vec\eta)\frac{\vec\sigma}{2}} \begin{pmatrix} \sigma^0 \\ \vec\sigma \end{pmatrix} e^{(-i\vec\theta+\vec\eta)\frac{\vec\sigma}{2}}\, \psi_R$$

consider separately inf. rotations ($\vec\theta$) and boosts ($\vec\eta$) and use Pauli-matrix (anti) commutation relation to show

rotations: $\psi_R^+ \begin{pmatrix} \sigma^0 \\ \vec\sigma \end{pmatrix} \psi_R \to \begin{pmatrix} \psi_R^+ \sigma^0 \psi_R \\ \psi_R^+ \vec\sigma \psi_R + \vec\theta \times \psi_R^+ \vec\sigma \psi_R \end{pmatrix}$

boosts: $\to \begin{pmatrix} \psi_R^+ \sigma^0 \psi_R + \vec\eta \cdot \psi_R^+ \vec\sigma \psi_R \\ \psi_R^+ \vec\sigma \psi_R + \vec\eta\, \psi_R^+ \sigma^0 \psi_R \end{pmatrix}$

these are trsf. properties of a 4-vector

similarly: $\psi_L^+ \bar\sigma^\mu \psi_L$ transforms as 4-vector

def: $\bar\sigma^\mu \equiv (1, -\vec\sigma)$

i.e. $\psi_L^+ \bar\sigma^\mu \psi_L \to \Lambda^\mu{}_\nu\, \psi_L^+ \bar\sigma^\nu \psi_L$

$$\Rightarrow \quad \mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \qquad (1)$$

is a (real and) L inv Lagrangian describing spin 1/2 !

We can rewrite this $\mathcal{L}$ by adding a total derivative

$$\mathcal{L} \to \mathcal{L} - \frac{i}{2}\partial_\mu\left(\psi_R^\dagger \sigma^\mu \psi_R + \psi_L^\dagger \bar{\sigma}^\mu \psi_L\right) \quad \text{to get}$$

$$\mathcal{L} = \frac{i}{2}\psi_R^\dagger \sigma^\mu \overleftrightarrow{\partial_\mu} \psi_R + \frac{i}{2}\psi_L^\dagger \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} \psi_L \qquad (2)$$

(where $f \overleftrightarrow{\partial_\mu} g \equiv f(\partial_\mu g) - (\partial_\mu f) g$)

Note: (1) and (2) are equivalent (same EoM)

energy-momentum tensor is different, conserved currents are different, but conserved charges are the same

<u>EoM</u>: $$\frac{\partial \mathcal{L}}{\partial \psi_{R/L}^\dagger} - \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_{R/L}^\dagger)}}_{} = 0$$

gives 0 if we take $\mathcal{L}$ from (1)

$\rightarrow$ Weyl equations
$$\boxed{\begin{aligned} i(\sigma^\mu \partial_\mu)\psi_R &= 0 \\ i(\bar{\sigma}^\mu \partial_\mu)\psi_L &= 0 \end{aligned}}$$

<u>solutions</u>: plane-wave sol. with positive energy $\psi_L(x) = u_L(p)\, e^{-ipx}$

$$\rightarrow \left(i\partial_t - i\vec{\sigma}\vec{\nabla}\right)\psi_L = 0 \quad \rightarrow \quad \left(E + \vec{\sigma}\cdot\vec{p}\right) u_L(p) = 0$$

$$\Rightarrow \quad \frac{\vec{\sigma}\cdot\vec{p}}{E}\, u_L = -u_L \quad \Rightarrow \quad \hat{p}\cdot\vec{J}\, u_L = -\frac{1}{2} u_L$$

$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\vec{J} = \frac{\vec{\sigma}}{2}$, $\hat{p} = \frac{\vec{p}}{E}$

helicity

for a right-handed positive energy solution $\psi_R(x) = u_R\, e^{-ipx}$

$$\to \quad \vec{p}\cdot\vec{J}\, u_R = +\tfrac{1}{2}\, u_R \qquad (\bar{\sigma}^\mu = (1,-\vec{\sigma}) \to \sigma^\mu = (1,+\vec{\sigma}))$$

massless Weyl spinors are eigenstates of helicity with $\pm 1/2$ eval

helicity is conserved in massless case, ψ_L and ψ_R do not mix

recall section 2.6!

adding a mass term

$m\, \psi_L^+ \psi_R + m\, \psi_R^+ \psi_L$ is also L inv!

$$\to \psi_L^+ D_L^+(\Lambda) D_R(\Lambda) \psi_R = \psi_L^+ \underbrace{\left(e^{(-i\vec{\theta}-\vec{\eta})\frac{\vec{\sigma}}{2}}\right)^+ \left(e^{(-i\vec{\theta}+\vec{\eta})\frac{\vec{\sigma}}{2}}\right)}_{=1} \psi_R = \psi_L^+ \psi_R$$

$$\mathcal{L} = i\, \psi_R^+ \sigma^\mu \partial_\mu \psi_R + i\, \psi_L^+ \bar{\sigma}^\mu \partial_\mu \psi_L - m\,(\psi_L^+ \psi_R + \psi_R^+ \psi_L)$$

is a L inv (and real, if $m \in \mathbb{R}$) Lagrangian for spin $\frac{1}{2}$

EoM:
$$\boxed{\begin{aligned} i\,(\sigma^\mu \partial_\mu)\, \psi_R &= m\, \psi_L \\ i\,(\bar{\sigma}^\mu \partial_\mu)\, \psi_L &= m\, \psi_R \end{aligned}}$$

$\left.\begin{array}{l}\text{Note: } \psi_L \,\&\, \psi_R \text{ mix!} \\ \text{coupled EoM}\end{array}\right\} \to \text{Dirac}$

before adding a mass term, we have 2 symmetries

$$\psi_L \to e^{i\theta_L} \psi_L \quad \text{and} \quad \psi_R \to e^{i\theta_R} \psi_R$$

with 2 conserved (Noether) currents

$$j_L^\mu = \psi_L^+ \bar{\sigma}^\mu \psi_L \quad \text{and} \quad j_R^\mu = \psi_R^+ \sigma^\mu \psi_R$$

after adding a mass term, we have only 1 symmetry left

$$\psi_{L/R} \to e^{i\theta} \psi_{L/R}$$

and 1 conserved current

3.2. The Dirac equation

Define Dirac spinor $\psi_\alpha = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, 4 components

Define γ-matrices: $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$ 4×4 matrices

i.e. $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ $\vec\gamma = \begin{pmatrix} 0 & \vec\sigma \\ -\vec\sigma & 0 \end{pmatrix}$ Chiral or Weyl representation

The "Weyl" Lagrangian can now be written as

$$\boxed{\mathcal{L} = \bar\Psi\,(i\partial_\mu\gamma^\mu - m)\,\Psi} = (\psi_L^+, \psi_R^+)\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} -m & i\partial_\mu\sigma^\mu \\ i\partial_\mu\bar\sigma^\mu & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

def Dirac adjoint $\bar\Psi \equiv \Psi^+\gamma^0$

use $\mathcal{L}' = (-i\partial_\mu\bar\Psi)\gamma^\mu\Psi - m\bar\Psi\Psi$

EoM: Dirac equation: $(i\partial_\mu\gamma^\mu - m\cdot\mathbb{1})\,\Psi = 0$

Notation: often use $\not{a} \equiv a_\mu\gamma^\mu = a^\mu\gamma_\mu$ $\rightarrow$ $\boxed{(i\not{\partial} - m)\,\Psi = 0}$

Note: The γ-matrices satisfy the following relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\cdot\mathbb{1} \qquad \left(\{a,b\} \equiv ab + ba\ ,\ \text{anticommutator}\right)$$

follows from properties of Pauli matrices

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2g^{\mu\nu}$$

Define $\gamma^5 = -\frac{i}{4!}\,\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$

$\Rightarrow$ $\frac{1}{2}(1-\gamma_5)\,\Psi = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\Psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$

$\frac{1}{2}(1+\gamma_5)\,\Psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$

Note $\{\gamma_5, \gamma_\mu\} = 0$ and $(\gamma_5)^2 = \mathbb{1}$

Furthermore $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^+$ (use explicit rep. to show)

The adjoint Dirac eq.

$$0 = [(i\partial\!\!\!/ - m)\psi]^+ = \psi^+(-i\overleftarrow{\partial}_\mu(\gamma^\mu)^+ - m) = \bar{\psi}\gamma^0(-i\partial_\mu\gamma^{\mu +} - m)$$

derivative acts on $\psi^+(x)$ $\qquad$ $\psi^+ = \psi^+(\gamma^0)^2 = \bar{\psi}\gamma^0$

now use $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^+ \quad \rightarrow \quad \gamma^0\gamma^{\mu +} = \gamma^\mu\gamma^0$

$$0 = \bar{\psi}(-i\overleftarrow{\partial}_\mu\gamma^\mu - m)\gamma^0 \quad \Rightarrow \quad \boxed{\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0}$$

can be obtained from $\mathcal{L} = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial \mathcal{L}}{\partial\psi} = (\partial_\mu\bar{\psi})\, i\gamma^\mu + m\bar{\psi} = 0$$

We can use different representations for γ matrices

$\gamma^\mu_{new} = U\gamma^\mu U^+$ (equivalent representations)

e.g. $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\Rightarrow$ Dirac representation $\quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$

Dirac equation in external el.mag. field A^μ (Vector field!)

$\bar{\psi}\gamma_\mu\psi$ transforms as a vector under LT (by construction from Weyl spinors!) $\quad \rightarrow \quad A^\mu\bar{\psi}\gamma_\mu\psi$ is L inv

$p^\mu = i\partial^\mu \quad \rightarrow \quad i\partial^\mu - eA^\mu$ minimal substitution

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi$$

3.3. Covariance of Dirac equation

The approach taken here is quite different from the usual "textbook" approach to the Dirac equation

[insist on linear equation consistent with KG eq

→ then study covariance! → exercise]

By construction $\bar{\Psi}\gamma^\mu\Psi$ transforms as a 4-vector!

We know how Ψ transforms in Weyl representation!

→ generalize to arbitrary rep. of Dirac matrices

$$\left.\begin{array}{l}\Psi'(x') = S(\Lambda)\,\Psi(x) \quad \text{with} \quad \Psi(x) = \begin{pmatrix} \Psi_L(x) \\ \Psi_R(x) \end{pmatrix} \\ S(\Lambda) = \begin{pmatrix} D_L(\Lambda) & 0 \\ 0 & D_R(\Lambda) \end{pmatrix} = \begin{pmatrix} e^{(-i\vec{\theta}-\vec{\eta})\vec{\sigma}/2} & \\ & e^{(-i\vec{\theta}+\vec{\eta})\vec{\sigma}/2} \end{pmatrix}\end{array}\right\} \text{Weyl rep.}$$

Claim: we can write $S(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}$ with $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

in Weyl rep.: $\sigma^{\mu\nu} = \frac{i}{2}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}$

$$\Rightarrow \sigma^{0j} = i\begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \qquad \sigma^{ij} = \frac{i}{2}\begin{pmatrix} -[\sigma^i, \sigma^j] & 0 \\ 0 & -[\sigma^i\sigma^j] \end{pmatrix} = \varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\Rightarrow -\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu} = -i\left(\sum_j \omega_{0j}\sigma^{0j} + \omega_{12}\sigma^{12} + \text{cycl.}\right) \qquad \left(\omega_{0j} = \eta_j \quad \omega_{12} = \theta_3 + \text{cycl.}\right)$$

$$= \begin{pmatrix} -\vec{\eta}\cdot\vec{\sigma} - i\vec{\theta}\vec{\sigma} & 0 \\ 0 & \vec{\eta}\vec{\sigma} - i\vec{\theta}\vec{\sigma} \end{pmatrix}$$

⇒ exponentiate to get desired result ☒

$$\Rightarrow \boxed{\begin{array}{l}\Psi \to S\cdot\Psi \\ \bar{\Psi} \to \bar{\Psi}S^{-1}\end{array}} \quad \text{and} \quad \bar{\Psi} = \Psi^\dagger\gamma^0 \to \Psi^\dagger S^\dagger\gamma^0 = \bar{\Psi}\gamma^0 S^\dagger\gamma^0 = \bar{\Psi}S^{-1}$$

exercise

We know $\bar{\Psi}(x)\gamma^\mu\Psi(x) \to \bar{\Psi}'(x')\gamma^\mu\Psi'(x')$

$$= \bar{\Psi}(x)\, S^{-1}\gamma^\mu S\, \Psi(x) \stackrel{!}{=} \Lambda^\mu{}_\nu\, \bar{\Psi}(x)\gamma^\nu\Psi(x)$$

Note: in an exercise, this will be derived again, requiring that Dirac eq. is covariant

writing $S = e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}$ we can work in any representation of the γ matrices

next deduce trsf. of Ψ under discrete transformations $\mathcal{P}, \mathcal{T}$ from trsf property of Weyl spinors.

Parity $\mathcal{P}$

recall $\vec{J} \to \vec{J}$, $\vec{K} \to -\vec{K}$ $\Rightarrow$ $\vec{\theta} \to \vec{\theta}$, $\vec{\eta} \to -\vec{\eta}$

$\Rightarrow$ $D_L = e^{(-i\vec{\theta}-\vec{\eta})\vec{\sigma}/2} \to D_R = e^{(-i\vec{\theta}+\vec{\eta})\vec{\sigma}/2}$ and $\psi_L \leftrightarrow \psi_R$

$\Rightarrow$ in Weyl: $\Psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \to \begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \gamma^0\Psi$

true in any rep of γ matrices

$$\Psi'(x') = \Psi'(t,\vec{x}') = \Psi(t,-\vec{x}) = e^{i\varphi}\gamma^0\Psi(x)$$

can add unobservable phase (usually $\varphi \to 0$)

or use $S^{-1}(P)\,\gamma^\mu\, S(P) = P^\mu{}_\nu\gamma^\nu = (\gamma^0, -\vec{\gamma})$

$$P^\mu{}_\nu = \begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$$

and $(\gamma^0)^{-1}\gamma^\mu\gamma^0 = \gamma^0\gamma^\mu\gamma^0 = (\gamma^0, -\vec{\gamma})$

Charge conjugation

Recall Section 2.4 where we defined charge conjugation

$\psi_L^c = i\sigma_2 \psi_L^*$ and $\psi_R^c = -i\sigma_2 \psi_R^*$

or $\Psi^c = -i\gamma^2 \Psi^* = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix}$

to understand the "name", consider Dirac eq. in ext. field

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\,\psi = 0 = (i\partial^\mu\gamma^\mu - eA^\mu\gamma_\mu - m)\,\psi$$

$$\Rightarrow \quad (-i\partial^\mu(\gamma_\mu)^* - eA^\mu(\gamma_\mu)^* - m)\,\psi^* = 0 \qquad (*)$$

next multiply from left by γ^2, using $\gamma^2(\gamma_\mu)^* = -\gamma_\mu\gamma^2$

$$\Rightarrow \quad (i\partial^\mu\gamma_\mu + eA^\mu\gamma_\mu - m)\,\gamma^2\psi^* = 0$$

thus $\Psi^c = -i\gamma^2\Psi^*$ satisfies $(i\partial\!\!\!/ + eA\!\!\!/ - m)\,\Psi^c = 0$

factor $-i$ phase convention — change of sign $e \to -e$

"charge conjugate" Dirac eq.

Bilinear covariants

Any 4×4 matrix can be decomposed into $\mathbb{1}, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu}$

d.o.f. $16 = 1 + 1 + 4 + 4 + 6$

$\bar\psi\psi$: scalar

$\bar\psi\gamma_5\psi$: pseudo scalar ($\to -\bar\psi\gamma_5\psi$ under P)

$\bar\psi\gamma_\mu\psi$: vector

$\bar\psi\gamma_5\gamma_\mu\psi$: axial vector

$\bar\psi\sigma^{\mu\nu}\psi$: tensor

$\to$ covariant building blocks with 2 spinors

34. Solutions to the Dirac equation

Free Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$

positive & negative energy solutions

$$\psi^+(x) = e^{-ipx}\, u(p)$$
$$\psi^-(x) = e^{+ipx}\, v(p)$$

u, v: 4 components
$\rightarrow$ Dirac spinors

Dirac eq. in momentum space:

$$\left.\begin{aligned}(p\!\!\!/ - m)\, u &= 0 \\ (p\!\!\!/ + m)\, v &= 0\end{aligned}\right\} \quad p\!\!\!/ \mp m = \begin{pmatrix} E \mp m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E \mp m \end{pmatrix} \quad (4\times 4 \text{ matrix})$$

in Dirac representation

$$p\!\!\!/ = p^\mu \gamma_\mu = E\gamma^0 - \vec{p}\cdot\vec{\gamma} \;;\quad m = m\cdot\mathbb{1} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

write $u(p) = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$ $\quad \chi, \varphi$: 2 components each

$$(p\!\!\!/ - m)\, u = 0 \quad \rightarrow \quad \chi = \frac{\vec{p}\cdot\vec{\sigma}}{E-m}\,\varphi \quad \text{and} \quad \varphi = \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\,\chi$$

<u>Solutions</u>

$$u(p,s) = \begin{pmatrix} \sqrt{E+m}\; \chi_s \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{\sqrt{E+m}}\; \chi_s \end{pmatrix} \quad \text{and similar} \quad v(p,s) = \begin{pmatrix} \dfrac{\vec{p}\cdot\vec{\sigma}}{\sqrt{E+m}}\; \chi_s \\ \sqrt{E+m}\; \chi_s \end{pmatrix}$$

$s = 1, 2$, labels two spin states $\quad \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>Note</u>: can also obtain these solutions starting from
$p^\mu = 0$, i.e. $\quad m\cdot\mathbb{1}\, u(p=0) = 0$

sol $\quad \sqrt{2m}\; \underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{u_{1,2}}, \underbrace{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{v_{1,2}}$ $\quad$ then boost to frame with momentum p^μ!

normalization

The adjoint eq: $[(\not p - m) u]^+ = 0 = u^+ \underset{1=\gamma_0^2}{\wedge} (p^\mu \gamma_\mu^+ - m)$ γ^0

$\Rightarrow \quad 0 = u^+ \gamma^0 (p^\mu \gamma^0 (\gamma_\mu)^+ \gamma_0 - m) = \bar{u} (p^\mu \gamma_\mu - m)$

$\Rightarrow$ the Dirac adjoint spinors $\bar{u} = u^+ \gamma_0$ and $\bar{v} = v^+ \gamma_0$

satisfy
$$\boxed{\begin{aligned} \bar{u}\,(\not p - m) &= 0 \\ \bar{v}\,(\not p - m) &= 0 \end{aligned}}$$

Normalization and orthogonality relations:

$$\bar{u}(p,s)\, u(p,r) = 2m\, \delta_{rs} \qquad \bar{u}(p,s)\, v(p,r) = 0$$
$$\bar{v}(p,s)\, v(p,r) = -2m\, \delta_{rs} \qquad \bar{v}(p,s)\, u(p,r) = 0$$

warning: sometimes different norm used, $u \to \frac{1}{2m} u$

Energy projection operators:

Define $\Lambda_\pm \equiv \dfrac{\pm \not p + m}{2m}$

$$\begin{cases} \Lambda_+ + \Lambda_- = 1 \\ \Lambda_+ \cdot \Lambda_- = \frac{1}{(2m)^2} (\not p + m)(\not p - m) \overset{*}{=} \frac{p^2 - m^2}{(2m)^2} = 0 \\ \Lambda_+^2 = \frac{1}{(2m)^2} (\not p + m)^2 = \frac{p^2 + m^2 + 2\not p m}{(2m)^2} = \frac{2m(\not p + m)}{(2m)^2} = \Lambda_+ \end{cases}$$

$\Lambda_+ u(p,s) = \frac{1}{2m} (\not p + m) u = \frac{1}{2m} (\underbrace{\not p - m}_{\downarrow 0} + 2m) u = u(p,s)$

$\Lambda_- u(p,s) = \frac{1}{2m} (-\not p + m) u = 0$

$\Lambda_+ v(p,s) = 0$ and $\Lambda_- v(p,s) = 0$

$\Lambda_\pm$ projects on positive / negative energy sol.

Check: $\sum_{s=1,2} u(p,s)\, \bar{u}(p,s) = \not p + m = 2m \Lambda_+$

$-\sum_{s=1,2} v(p,s)\, \bar{v}(p,s) = -\not p + m = 2m \Lambda_-$

$*$ $\not p \not p = p_\mu p_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} = \frac{1}{2} p_\mu p_\nu\, 2 g^{\mu\nu} = p^2 \cdot 1$