Exercise 1. ***Exact solution of the Ising chain***

In this exercise we will investigate the physics of one of the few *exactly solvable interacting* models, the one-dimensional Ising model (Ising chain). Consider a chain of $N+1$ Ising-spins with free ends and nearest neighbor coupling $-J$ ($J > 0$ for ferromagnetic coupling)

$$\mathcal{H}_{N+1} = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1}, \quad \sigma_i = \pm 1. \tag{1}$$

In this exercise we will be interested in the thermodynamic limit of this system, i.e. we assume N to be very large.

(a) Compute the partition function Z_{N+1} using a recursive procedure.

(b) Find expressions for the free energy and entropy, as well as for the internal energy and heat capacity. Compare your results to the ideal paramagnet.

(c) Calculate the magnetization density $m = \langle \sigma_j \rangle$ where the spin σ_j is far away from the ends. Which symmetries does the system exhibit? Interpret you result in terms of symmetry arguments.

(d) Show that the *spin correlation function* $\Gamma_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$ decays exponentially with increasing distance $|j-i|$ on the scale of the so-called *correlation length* ξ, i.e. $\Gamma_{ij} \sim \mathrm{e}^{-|j-i|/\xi}$. Show that $\xi = -[\log(\tanh \beta J)]^{-1}$ and interpret your result in the limit $T \to 0$.

(e) Calculate the magnetic susceptibility in zero magnetic field using the fluctuation-dissipation relation of the form

$$\frac{\chi(T)}{N} = \frac{1}{k_{\mathrm{B}} T} \sum_{j=-N/2}^{N/2} \Gamma_{0j}, \tag{2}$$

in the thermodynamic limit, $N \to \infty$. For simplicity we assume N to be even. Note that $\chi(T)$ is defined to be extensive, such that we obtain the intensive quantity by normalization with N.

(f) Approximate $1/\chi(T)$ up to first order in $2\beta J$ in the high-temperature limit ($\beta \to 0$). Use this result to calculate the *Weiss temperature* Θ_{W}, which is defined by $1/\chi(\Theta_{\mathrm{W}}) = 0$.